Extremal Combinatorial Problems

Dániel Gerbner

Ph.D Dissertation

Supervisor: Gyula O.H. Katona

Informatics Ph.D School

Dr. János Demetrovics

Foundation and Methods in Informatics Ph.D Program

Dr. János Demetrovics

Eötvös Loránd University

Faculty of Informatics

Budapest, 2009

Contents

Chapter 1

Introduction

1.1 Background

Most of the results of the present thesis belong to the theory of extremal set systems. These are theoretical results with close connection to theoretical computer science. The last chapter however deals with problems in theoretical computer sciences, namely in the theory of sorting.

One of the most typical questions in the theory of extremal set families is the following: We are given a family of subsets of an n-element underlying set, satisfying some properties. How many sets can that family contain?

The best-known result of this area is Sperner's theorem, which says that an inclusion-free family can contain at most $\binom{n}{\lceil n/2 \rceil}$ members. The celebrated theorem of Erdős, Ko and Rado claims if $k \le n/2$, a family of pairwise intersecting k-element subsets can have at most $\binom{n-1}{k-1}$ members.

Let us introduce the most important notations.

Let $X = [n] = \{1, \ldots, n\}$. The power set of X is denoted by 2^X. A subset of 2^X is called a **family**. A family is k-**uniform** if its members are all of size k.

The **complement** of a subset A of X will be denoted by $\overline{A}$ (the universal set will always be clear from the context). The collection of all k element subsets of the set X will be denoted by $\binom{X}{k}$. For a set family $\mathcal{F} \subseteq 2^X$ let $\mathrm{co}(\mathcal{F}) = \{F \subseteq [n] : \overline{F} \in \mathcal{F}\}$.

A **chain** is a family $\mathcal{L} = \{L_1, L_2, \ldots, L_i\}$ such that $L_1 \subset L_2 \subset \ldots \subset L_i$. A **full chain** $\mathcal{L}$ is a chain of length $n+1$, i.e. $\mathcal{L} = \{L_0, L_1, \ldots, L_n\}$ such that $L_0 \subset L_1 \subset$

$\dots \subset L_n$. $|L_i| = i$ easily follows. $\mathcal{K} = \mathcal{L} \cup \mathrm{co}(\mathcal{L})$ is a **full complement chain-pair**, or briefly **chain-pair**.

A family $\mathcal{F} \subset 2^X$ will be called a **Sperner family** if it is inclusion-free, i.e. there are no distinct $F, F' \in \mathcal{F}$ such that $F \subset F'$.

A family $\mathcal{F} \subseteq 2^{[n]}$ will be called k-**Sperner** if it contains no chain $F_0 \subsetneq F_1 \subsetneq \dots F_k$ of $k+1$ different sets. For $k = 1$ this is the usual notion of Sperner families, for $k = 0$ the only 0-Sperner family is $\mathcal{F} = \emptyset$.

A family $\mathcal{F} \subseteq 2^{[n]}$ will be called t-**intersecting** if for any $F, F' \in \mathcal{F}$ $|F \cap F'| \geq t$. A 1-intersecting family will be called **intersecting**, the complement family $\overline{\mathcal{F}}$ of an intersecting family $\mathcal{F}$ is **co-intersecting**. It means a family $\mathcal{F}$ is co-intersecting if and only if for any $F, F' \in \mathcal{F}$ these two sets do not cover the whole underlying set, i.e. $|F \cup F'| < n$.

For a $w : \{0, \dots, n\} \to \mathbb{R}$ **weight function** let $w(A) = w(|A|)$ for an $A \subset X$ and $w(\mathcal{F}) = \sum_{F \in \mathcal{F}} w(F)$ for a family $\mathcal{F}$. Usually we are given a class of families and a weight function, and we are interested in a family of the maximal weight. However we will see we can deal with every weight function at the same time.

Similarly, for a function $\alpha : [n] \to [n]$ let $\alpha(A) = \{\alpha(x) : x \in A\}$ and $\alpha(\mathcal{F}) = \{\alpha(A) : A \in \mathcal{F}\}$.

If $\mathcal{F}$ is a family of k element sets, then the **shadow** of $\mathcal{F}$ is $\triangle\mathcal{F} = \{A \subseteq [n]\colon |A| = k-1$, there exists $F \in \mathcal{F}$ such that $A \subseteq F\}$. The **shade** of $\mathcal{F}$ is $\nabla\mathcal{F} = \{A \subseteq [n]\colon |A| = k+1$, there exists $F \in \mathcal{F}$ such that $F \subseteq A\}$.

The **upset** of a family $\mathcal{F}$ is $\mathcal{U}(\mathcal{F}) = \{G \subseteq X : \exists F \in \mathcal{F} \text{ such that } F \subseteq G\}$ and the **downset** of $\mathcal{F}$ is $\mathcal{D}(\mathcal{F}) = \{G \subseteq X : \exists F \in \mathcal{F} \text{ such that } F \supseteq G\}$.

A class $\mathbf{A}$ of families is **upward (downward) closed** if $\mathcal{F} \in \mathbf{A}$ implies $\mathcal{U}(\mathcal{F}) \in \mathbf{A}$ ($\mathcal{D}(\mathcal{F}) \in \mathbf{A}$). Clearly the class of t-intersecting (t-co-intersecting) families is upward (downward) closed.

For a family $\mathcal{F}$ let $\mathrm{conv}(\mathcal{F}) = \{G \subseteq X : \exists F, F' \in \mathcal{F}(F \subseteq G \subseteq F')\}$ denote its **convex closure**. $\mathcal{F}$ is said to be **convex** if $\mathcal{F} = \mathrm{conv}(\mathcal{F})$. A class of families $\mathbf{A}$ is said to be **convex closed** if $\mathcal{F} \in \mathbf{A}$ implies $\mathrm{conv}(\mathcal{F}) \in \mathbf{A}$. The basic example for a convex closed set is the class of intersecting and co-intersecting families.

A **partially ordered set** (briefly **poset**) is an ordered pair $(P, \prec)$, where $\prec$ is a transitive, irreflexive and antisymmetric relation on P.

Our main example is the **Boolean poset** $(2^{[n]}, \subset)$. Also any part $(\mathcal{H}, \subset)$ of the Boolean poset can be considered as a poset. Especially interesting for us is if $\mathcal{H}$ is a full chain, or a chain-pair, or the family of the intervals (defined in the next section).

The **poset of subspaces** is $(L(n,q), \subset)$, where $L(n,q)$ is the set of the subspaces of an n-dimensional vector space over a q-element field. This poset will be studied in Chapter 5.

The **rank function** of a poset is a function $r : P \to \mathbb{N}$ such that if x covers y (i.e. $y \prec x$ and there is no $z \in P$ such that $y \prec z \prec x$), then $r(x) = r(y) + 1$ and there is at least one element x with $r(x) = 0$.

1.2 Permutation method

Next we describe our main method, the permutation method. It was first used by Lubell ([28]). We present his proof of the famous LYM-inequality ([28], [29], [40]).

Theorem 1 (Sperner) *If $\mathcal{F}$ is a Sperner-family, then $|\mathcal{F}| \le \binom{n}{\lfloor n/2 \rfloor}$.*

Sperner ([38]) proved it via exchanging members of $\mathcal{F}$ to other sets such that the whole family remains a Sperner family, but it is usually proved due to the LYM-inequality.

Theorem 2 (LYM-inequality) *If $\mathcal{F}$ is a Sperner-family, then*

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{F}} \le 1.$$

Proof. Any set of size i is contained in $i!(n-i)!$ full chains. The number of pairs $(\mathcal{C}, F)$ such that $\mathcal{C}$ is a full chain and F is a member of $\mathcal{F} \cap \mathcal{C}$ is $\sum_{F \in \mathcal{F}} |F|!(n-|F|)!$. On the other hand there are $n!$ full chains and each of them can contain at most one member of $\mathcal{F}$. Hence

$$\sum_{F \in \mathcal{F}} |F|!(n-|F|)! \le n!,$$

which is equivalent to the statement. ■

Proof of Theorem 1. $\binom{n}{|F|}$ is at most $\binom{n}{\lfloor n/2 \rfloor}$ hence $1/\binom{n}{|F|}$ is at least $1/\binom{n}{\lfloor n/2 \rfloor}$. It follows that

$$1 \geq \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{F}} \geq \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{\lfloor n/2 \rfloor}},$$

so $|\mathcal{F}| = \sum_{F \in \mathcal{F}} 1 \leq \binom{n}{\lfloor n/2 \rfloor}$.■

Theorem 3 (Erdős, [12]) *Let $\mathcal{F}$ be a k-Sperner family of subsets of an n-element set. Then*

$$|\mathcal{F}| \leq \sum_{i=\lfloor (n-k+1)/2 \rfloor}^{\lfloor (n+k-1)/2 \rfloor} \binom{n}{i}.$$

Proof. A k-Sperner family $\mathcal{F}$ is the union of k disjoint Sperner family: let $\mathcal{F}^1$ be the subfamily of the maximal members of $\mathcal{F}$, $\mathcal{F}^2$ be the family of maximal members of $\mathcal{F} \setminus \mathcal{F}^1$, and so on, $\mathcal{F}^i$ be the family of maximal members of $\mathcal{F} \setminus \cup_{j=1}^{i} \mathcal{F}^j$. It is easy to see these are all Sperner families and if $i > k$ then $\mathcal{F}^i$ is empty. Hence it follows

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{F}} \leq k.$$

The size of $\mathcal{F}$ can be the largest if we choose the sets with the smallest weight $\frac{1}{\binom{n}{F}}$. These are the sets on the middle levels , which proves the theorem. ■

We will use **the method of cyclic permutations** developed by Gyula O.H. Katona in [26] (for a survey on this topic see [27]). First of all let us introduce some terminology.

Let the elements of the set $[n]$ be placed around a circle such that $i+1$ is next to i for all $i = 1, 2, \ldots, n-1$ and 1 is next to n in clockwise direction: we will also say that $i+1$ is to the right of i. (We consider these numbers mod n). Elements next to each other will be called **consecutive**. A set of consecutive elements will be called an **interval**. Denote the interval of elements between a and b by $[a, b]$ (endpoints included): this is the set of elements $a, a+1, \ldots, b$. The family of all intervals on the circle will be denoted by $\mathcal{H}$. For a given set F of size k there are $k!(n-k)!$ cyclic permutations which map F to an interval.

We will say a family is "on the circle", to mean its members are intervals. If $\mathcal{F}$ is a k-uniform family on the circle, the shadow and shade can be analogously defined but considering only intervals. More precisely let $\nabla_{\text{int}}\mathcal{F} := \{A \in \mathcal{H}\colon |A| = k+1$, there exists $F \in \mathcal{F}$ such that $F \subseteq A\}$ and $\Delta_{\text{int}}\mathcal{F} = \{A \in \mathcal{H}\colon |A| = k-1$, there exists $F \in \mathcal{F}$ such that $A \subseteq F\}$.

Theorem 4 (Erdős, Ko, Rado) *If k and n are natural numbers with $k \leq n/2$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is an intersecting family then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.*

Lemma 5 (G.O.H. Katona) *If $k \leq n/2$ and $\mathcal{G}$ is an intersecting family of intervals of size k then $|\mathcal{G}| \leq k$.*

Proof. We can assume $F = [1, k]$ is a member of $\mathcal{G}$. All the other members of $\mathcal{G}$ are of form $[x, i]$ for $1 \leq i \leq k-1$ and appropriate x or $[j, y]$ for $2 \leq j \leq k$ and appropriate y. These are $2k-2$ additional possible intervals, but only half of them can belong to $\mathcal{G}$. For any i, the intervals $[x, i]$ and $[i+1, y]$ do not intersect because of the property $k \leq n/2$. Hence there are at most $k-1$ additional members and F in $\mathcal{G}$, which proves the lemma. ■

Proof of Theorem 4. Consider the pairs (α, F) where α is a cyclic permutation, F is an element of $\mathcal{F}$ and $\alpha(F)$ is an interval. For a given $F \in \mathcal{F}$ there are $k!(n-k)!$ cyclic permutations such that $\alpha(F)$ is an interval, hence the number of pairs is exactly $|\mathcal{F}|k!(n-k)!$. On the other hand for a given α there are at most k intersecting k-element intervals because of Lemma 5. Hence the number of pairs is at most $k(n-1)!$. So $|\mathcal{F}|k!(n-k)! \leq k(n-1)!$, the theorem follows. ■

In the previous proofs we reduced the problem to a simpler structure instead of the Boolean poset. In the first proof we considered every full chain, in the last proof we used cyclic permutations, but the common generalization shows the similarity.

Let $\mathcal{H}$ be a family. Let h_i denote the size of $\mathcal{H}_i = \mathcal{H} \cap \binom{[n]}{i}$. Let $\mathbf{A}$ be a class of families and $\mathcal{F}$ be a member of $\mathbf{A}$. We are interested in the maximal size (or weight) of $\mathcal{F}$. Consider the pairs (α, F), where α is a permutation, $F \in \mathcal{F}$ and $\alpha(F) \in \mathcal{H}$. For a given $F \in \mathcal{F}$ and a given $H \in \mathcal{H}$ of the same size there are $|F|!(n-|F|)!$ permutations that map F to H. Hence $h_{|F|}|F|!(n-|F|)!$ permutations map F to an element of $\mathcal{H}$, hence there are $\sum_{F \in \mathcal{F}} h_{|F|}|F|!(n-|F|)!$ such pairs. On the other hand there are $n!$ permutations. If we know $|\alpha(\mathcal{F}) \cap \mathcal{H}| \leq x$ for any α, then

$$\sum_{F \in \mathcal{F}} h_{|F|}|F|!(n-|F|)! \leq n!x.$$

If $\mathcal{F}$ is a Sperner family and $\mathcal{H}$ is a chain (hence $h_i = 0$ for every i), then $|\alpha(\mathcal{F}) \cap \mathcal{H}| \leq 1$ is trivial, and gives the LYM-inequality. If $\mathcal{F}$ is a k-uniform intersecting family with $0 < k \leq n/2$ and $\mathcal{H}$ is the family of intervals (so $h_k = n$), then $|\alpha(\mathcal{F}) \cap \mathcal{H}| \leq k$ by Lemma 5, and it gives Theorem 4.

Suppose we are given a weight function w, and we know $w(\alpha(\mathcal{F}) \cap \mathcal{H}) \leq x$. Let us count the weight of F for all possible pairs (α, F), where α is a permutation, $F \in \mathcal{F}$ and $\alpha(F) \in \mathcal{H}$. Clearly it is $\sum_{F \in \mathcal{F}} h_{|F|} w(|F|)|F|!(n-|F|)!$, and on the other hand it is at most $n!x$. If we denote $h_i w(i) i!(n-i)!$ by $w'(i)$, we can bound $w'(\mathcal{F})$.

We can even choose w' to be constant. Suppose we are given a k-Sperner family $\mathcal{F}$, let $\mathcal{H}$ be a full chain and $w(i) = \binom{n}{i}/n!$, then $\alpha(\mathcal{F}) \cap \mathcal{H}$ can contain at most k elements. The weight of them are the largest if they are the middle ones. Hence we can find $w(\alpha(\mathcal{F}) \cap \mathcal{H}) \leq x = \sum_{i=\lfloor (n-k+1)/2 \rfloor}^{\lfloor (n+k-1)/2 \rfloor} \binom{n}{i}/n!$. Clearly $w'(i) = 1$ for any i, so Theorem 3 easily follows.

Suppose we are given a class of families $\mathbf{A}$, and we are interested in its member $\mathcal{F}$ of maximal size or weight. For a tight result $w(\alpha(\mathcal{F}) \cap \mathcal{H}) \leq x$ should be tight for every permutation α. This can happen only if both $\mathcal{F}$ and $\mathcal{H}$ are very symmetric. So far there have been found only three families which give tight results for nontrivial problems, the full chain, the family of intervals and the chain-pair.

1.3 Profile polytopes

Let f_i denote the size of the subfamily of the i-element subsets in $\mathcal{F}$: $f_i = |\{F : F \in \mathcal{F}, |F| = i\}|$. The vector $\mathbf{p}(\mathcal{F}) = (f_0, f_1, \ldots, f_n)$ in the $(n+1)$-dimensional space $\mathbb{R}^{n+1}$ is called the **profile** of $\mathcal{F}$. The vector $\mathbf{p}_0(\mathcal{F}) = (f_1, f_2, \ldots, f_{n-1})$ is called the **reduced profile** of $\mathcal{F}$.

If Λ is a finite set in $\mathbb{R}^d$, its **convex hull** $\mathrm{conv}(\Lambda)$ is the set of all convex combinations of the elements of Λ. A point of Λ is an **extreme point** if it is not a convex combination of other points of Λ. It is easy to see that the convex hull of a set is equal to the convex hull of the extreme points of the set.

We call an extreme point $\mathbf{v}$ of a set Λ **essential** if there is no other point $\mathbf{u} \in \Lambda$ with $\mathbf{v} \leq \mathbf{u}$ (it denotes $v_i \leq u_i$ for every i). We say that a $\Gamma = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ set of vectors **dominates** a Λ set of vectors if for any $\mathbf{v} \in \Lambda$ there are constants $\lambda_1, \ldots \lambda_m \geq 0$, $\sum_{i=1}^m \lambda_i \leq 1$ satisfying $\mathbf{v} \leq \sum_{i=1}^m \lambda_i \mathbf{v}_i$. We say that $\mathbf{A}$ is **hereditary** if $\mathcal{F} \subseteq \mathcal{F}' \in \mathbf{A}$ implies $\mathcal{F} \in \mathbf{A}$.

Let $\mathbf{A}$ be a class of families of subsets of $[n]$. Let $\mu(\mathbf{A})$ denote the set of all profile vectors of families in $\mathbf{A}$, $\mathcal{E}(\mathbf{A})$ the extreme points of $\operatorname{conv}(\mu(\mathbf{A}))$ (we simply call them the extreme points of $\mathbf{A}$) and $E(\mathbf{A})$ the families from $\mathbf{A}$ with profile in $\mathcal{E}(\mathbf{A})$. Let furthermore $\mathcal{E}^*(\mathbf{A})$ denote the essential extreme points and $E^*(\mathbf{A})$ the corresponding families. The profile polytope of $\mathbf{A}$ is $\operatorname{conv}(\mu(\mathbf{A}))$

Proposition 6 . *If $\mathbf{A}$ is hereditary, then*

(1) any element of $\mathcal{E}(\mathbf{A})$ can be obtained by changing some coordinates of an element of $\mathcal{E}^*(\mathbf{A})$ to zero.

(2) If $\Gamma \subset \mu(\mathbf{A})$ dominates $\mu(\mathbf{A})$ then $\mathcal{E}^*(\mathbf{A}) \subset \Gamma$.

For the reduced profiles the analogous statement is true. In both cases the proofs are trivial (see [15]).

(1) of Proposition 6 shows it is enough to find the essential extreme points, and (2) shows they can be found much easier. We do not have to deal with equalities, only with inequalities.

On the other hand, suppose we are given a weight function $w : \{0, \dots, n\} \to \mathbb{R}$, and the weight of a family $\mathcal{F}$ is defined to be $\sum_{F\in\mathcal{F}} w(|F|)$, which is equal to $\sum_{i=0}^{n} w(i) f_i$. Usually we are interested in the maximum of the weight of the families in a class $\mathbf{A}$. So we want to maximize this sum, i.e. find a family $\mathcal{F}_0 \in \mathbf{A}$ and an inequality $\sum_{i=0}^{n} w(i) f_i = w(\mathcal{F}) \le w(\mathcal{F}_0) = c$. This is a linear inequality, and it is always maximized in an extreme point (if the weight function is positive, it is maximized in an essential extreme point). We usually want to find the maximum weight, but conversely, it can help us to determine the extreme points. If we find a set of profile vectors, such that for every weight an element of this set gives the maximum, then they contain the extreme points. Moreover, if we find a set of profile vectors, such that for every positive weight an element of it gives the maximum, they contain the essential extreme points. This approach will later be used in Chapter 3.

The profile polytopes were introduced by P.L. Erdős, P. Frankl and G.O.H. Katona in [14], a survey on this topic can be found in [9].

Let us introduce the following notations: suppose $A, B \subseteq \{0, \dots, n\}$ be disjoint sets. Then

$$(\mathbf{x}_{A,B})_i = \begin{cases} \binom{n}{i} & \text{if } i \in A \\ \binom{n-1}{i-1} & \text{if } i \leq n/2 \text{ and } i \in B \\ \binom{n-1}{i} & \text{if } i > n/2 \text{ and } i \in B \\ 0 & \text{otherwise.} \end{cases}$$

Here $\binom{n-1}{-1} := 1$. We also define

$$(\mathbf{u}_{A,B})_i = \begin{cases} 1 & \text{if } i = 0, n \text{ and } i \in A \cup B \\ n & \text{if } i \neq 0, n \text{ and } i \in A \\ i & \text{if } i \neq 0,\ i \leq n/2 \text{ and } i \in B \\ n-i & \text{if } i \neq n,\ i > n/2 \text{ and } i \in B \\ 0 & \text{otherwise.} \end{cases}$$

The main example for the family with profile vector $\mathbf{x}_{A,B}$ is the following: $\mathcal{F}_{A,B} = \mathcal{F}_A \cup \mathcal{F}'_B$ where $\mathcal{F}_A = \{F \subset [n] : |F| = i \text{ for some } i \in A\}$ and $\mathcal{F}'_B = \{F \subset [n] : 1 \in F \text{ and } |F| = j \text{ for a } j \in B, j \leq n/2\} \cup \{F \subset [n] : 1 \notin F \text{ and } |F| = j \text{ for a } j \in B, j > n/2\}$. Let $\mathcal{G}_{A,B} = \mathcal{F}_{A,B} \cap \mathcal{H}$, where $\mathcal{H}$ is the family of intervals, defined in the Section 2. The profile vector of $\mathcal{G}_{A,B}$ is $\mathbf{u}_{A,B}$.

We show one of the simplest examples of determining the extreme points of a class of families.

Theorem 7 ([14]) *Let* $\mathbf{A}$ *be the class of Sperner families. Then the extreme points of* $\mathbf{A}$ *are the vectors* $\mathbf{x}_{A,\emptyset}$*, where* $|A| \leq 1$.

Proof. Clearly $\mathbf{A}$ is hereditary. Let $\mathcal{F}$ be a Sperner family and $\mathbf{f} = (f_0, \ldots, f_n)$ be its profile vector. Let $\lambda_i = f_i / \binom{n}{i}$, then

$$\mathbf{f} = \sum_{i=0}^{n} \lambda_i \mathbf{x}_{\{i\},\emptyset}.$$

By the LYM-inequality $\sum_{i=0}^{n} \lambda_i \leq 1$, thus we are done by Proposition 6. ■

1.4 Profile polytopes and the permutation method

Profile polytopes are usually determined using the cycle method. In the very first paper of the area ([14]) P.L. Erdős, Frankl and Katona proved some inequalities about the intersecting Sperner families using the cycle method, which gave linear inequalities of the profile vectors. They gave a list of profile vectors and proved that if a set of vectors contains these vectors and satisfies those inequalities, then its extreme points are the given vectors. However, in their next paper ([15]) they showed how the profile vectors of families of intervals can be used directly to determine the extreme points in the general case.

Let us revisit the cycle method.

It has been mentioned, that if we have an upper bound for the weight of every $\alpha(\mathcal{F}) \cap \mathcal{H}$, where $\mathcal{F}$ is an element of a hereditary $\mathbf{A}$ and $\mathcal{H}$ is the family of the intervals, then we can get an upper bound for (an other) weight of the elements of $\mathbf{A}$. If we have upper bounds for every weight (on the circle), then we get upper bounds for every weight.

If $\mathbf{v} = (v_0, v_1, \ldots, v_n)$ then let

$$T(\mathbf{v}) = \left(v_0, v_1\binom{n}{1}/n, v_2\binom{n}{2}/n, \ldots, v_{n-1}\binom{n}{n-1}/n, v_n\right).$$

Theorem 8 ([15]) *If $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are the extreme points of $\mu(\mathbf{A}_\alpha)$ for any given cyclic permutation α then*

$$\mu(\mathbf{A}) \subseteq \operatorname{conv}\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_m)\}.$$

This theorem is really useful if $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_m) \in \mu(\mathbf{A})$ holds. (This can be easily checked.) Then $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_m)$ are the extreme points of $\mathbf{A}$. This theorem is a special case of Corollary 10 below.

Now let $\mathcal{H}$ be an arbitrary family which contains an i-element set for every $i \leq n$. Let

$$T_{\mathcal{H}}(\mathbf{v}) = \left(v_0/h_0, v_1\binom{n}{1}/h_1, v_2\binom{n}{2}/h_2, \ldots, v_{n-1}\binom{n}{n-1}/h_{n-1}, v_n/h_n\right).$$

Theorem 9 *Let $\mathcal{F}$ be a family. Suppose that the profile vector of $\alpha(\mathcal{F}) \cap \mathcal{H}$ is in the convex hull of $\mathbf{v}_1, \ldots, \mathbf{v}_m$ for every permutation α. Then the profile of $\mathcal{F}$ is in the convex hull of $T_{\mathcal{H}}(\mathbf{v}_1), \ldots, T_{\mathcal{H}}(\mathbf{v}_m)$.*

Proof. The profile of $\mathcal{F}$ is $\mathbf{p}(\mathcal{F}) = (f_0, \ldots, f_n)$. Let $\mathbf{v}_\alpha$ be the profile vector of $\alpha(F) \cap \mathcal{H}$. Then $\mathbf{v}_\alpha = \sum_{i=1}^m \lambda_i \mathbf{v}_i$, where $\sum_{i=1}^m \lambda_i = 1$. We introduce a vector-valued weight function

$$(\mathbf{w}(k))_i = \begin{cases} 1/n! & \text{if } i = k \\ 0 & \text{otherwise.} \end{cases}$$

We count the sum $\sum \mathbf{w}(|F|)$ for all pairs $(\alpha, \mathcal{F})$ where α is a permutation, $F \in \mathcal{F}$ and $\alpha(F) \in \mathcal{H}$. There is a convex combination $\sum_{i=1}^m \lambda_i(\alpha)\mathbf{v}_i = \mathbf{p}(\alpha(F) \cap \mathcal{H})$ for every α ($\sum_{i=1}^m \lambda_i(\alpha) = 1$, $\lambda_i(\alpha) \geq 0$).

Hence

$$\sum_{\alpha,F} \mathbf{w}(|F|) = \sum_{\alpha} \sum_{F} \mathbf{w}(|F|) = \sum_{\alpha} \frac{1}{n!} \sum_{i=1}^{m} \lambda_i(\alpha)\mathbf{v}_i = \sum_{i=1}^{m} \frac{1}{n!} (\sum_{\alpha} \lambda_i(\alpha))\mathbf{v}_i$$

where $\sum_{i=1}^m \frac{1}{n!} \sum_\alpha \lambda_i(\alpha) = 1$. Thus the above sum is a convex combination of $\mathbf{v}_1, \ldots, \mathbf{v}_m$.

On the other hand

$$\sum_{\alpha,F} \mathbf{w}(|F|) = \sum_{F} \sum_{\alpha} \mathbf{w}(|F|) = \sum_{F} |F|!(n - |F|)! h_{|F|} \mathbf{w}(|F|) = T_{\mathcal{H}}(\mathbf{p}(\mathcal{F})). \blacksquare$$

This proof follows the proof of Theorem 8 in [15].

Corollary 10 *Let* $\mathbf{A}$ *be a class of families. Suppose that the profile vector of* $\alpha(F) \cap \mathcal{H}$ *is in the convex hull of* $\mathbf{v}_1, \ldots, \mathbf{v}_m$ *for every permutation* α *and every* $\mathcal{F} \in \mathbf{A}$. *Then the profile polytope of* $\mathbf{A}$ *is in the convex hull of* $T_{\mathcal{H}}(\mathbf{v}_1), \ldots, T_{\mathcal{H}}(\mathbf{v}_m)$.

1.4.1 Reduction to the chain

In this subsection we consider some problems which can be solved using reduction to a chain. The first of them is from our joint paper ([22]) with Balázs Patkós. We have to mention that Sali in [36] uses arguments involving reduction to a full chain.

Theorem 11 *The extreme points of the convex hull of the profile vectors of convex families are the following: the all zero vector and for all* $0 \leq i \leq j \leq n$ *the vectors* $\mathbf{x}_{A_{i,j},\emptyset}$ *where* $A_{i,j} = \{i, i+1, \ldots, j\}$.

Proof. The vector $\mathbf{x}_{A_{i,j},\emptyset}$ is the profile of the family $\mathcal{F}_{i,j} = \{F \subseteq [n] : i \leq |F| \leq j\}$, which is convex.

On a chain any convex family must consist of some consecutive subsets of the chain. The theorem follows now from Corollary 10. ■

Note that the set of convex families is not hereditary, therefore the extreme points need not be the ones obtained from the essential extreme points (in this case there is only one such, the profile of $2^{[n]}$) by changing some of the non-zero components to zero. Theorem 11 shows they are indeed not those vectors.

Theorem 12 *The extreme points of the profile polytope of k-Sperner families are the vectors $x_{A,\emptyset}$ where $|A| \leq k$.*

This is a result of P.L. Erdős, P. Frankl and G.O.H. Katona [15]. They proved it by reducing to the circle instead of the chain.

Proof. It is trivial to see that these vectors are profiles of the corresponding levels, and they are convex independent.

A k-Sperner family on a full chain consists of at most k sets, therefore its profile vector have ones in at most k components. All these vectors are convex independent. Therefore they form the extreme points of the convex hull of the profile polytope on the chain, and Corollary 10 implies now Theorem 12. ■

1.4.2 Reduction to a chain-pair

In Section 3.2 we will determine the profile polytope of the set of complement-free k-Sperner families using reduction to the circle. In the case n is odd the much easier reduction to the chain-pair also works.

Theorem 13 *Let $n = 2m + 1$. Then the extreme points of the profile polytope of the set of complement-free k-Sperner families are the vectors $x_{A,\emptyset}$ where $|A| \leq k$ and $i \in A$ implies $n - i \notin A$.*

Proof. By Corollary 10, it is enough to prove the following

Lemma 14 *If $n = 2m + 1$, then the extreme points of the profile polytope of complement-free k-Sperner families on a pair of maximal complement chains are the vectors with at most k non-zero components, where all the non-zero components are 2 (except for the first or the last component, if one of them is non-zero, it equals 1), and if the ith component is non-zero, then the $(n - i)$th component is zero.*

Proof of Lemma 14. If the non-zero components of such a vector are $i_1, i_2, ..., i_z$ (satisfying the condition of the lemma), then the sets in the two chains with cardinality α_i for some $i = 1, ..., z$ form a complement-free k-Sperner family with the vector as profile.

Now let $\mathcal{F}$ be a complement-free k-Sperner family on a pair of complement chains $\mathbf{C}_1, \mathbf{C}_2$ with profile vector f. Let α be the set of indices of the non-zero components of f. Partition α into three subsets. Let CL (complete levels) denote the indices α_i with $f_{\alpha_i} = 2$ and furthermore CL contains 0 (n) provided $f_0 = 1$ $(f_n = 1)$. Let furthermore CP (complementing pairs) denote the indices $\alpha_i \in \alpha$ with $n - \alpha_i \in \alpha$, and let $R = \alpha \setminus (CL \cup CP)$. Note that $CP \cap CL = \emptyset$, for otherwise $\mathcal{F}$ would not be complement-free. Now form two subsets α^1, α^2 of α in the following way. Put all indices in CL into both α^1 and α^2. For all pairs of indices $i, n - i$ in CP (note that these are really pairs, for n is odd) put one of the indices into α^1 and the other into α^2. Finally, choose α^1 or α^2 for all indices of R in such a way, that $|\alpha^1| \le k$ and $|\alpha^2| \le k$ hold. (This is possible, for $\mathcal{F}$ is k-Sperner, therefore $|\alpha| \le 2k$.) Now let f^i, $i = 1, 2$ the following vectors.

$$f^i_j = \begin{cases} 2 \quad (1) \text{ if } \ j \neq 0, n \quad (j = 0, n) \quad j \in \alpha^i \\ 0 \qquad\qquad\qquad \text{otherwise.} \end{cases} \tag{1.1}$$

By the facts that both f^is are of the form of the statement of the lemma and $f = \frac{1}{2} f^1 + \frac{1}{2} f^2$, furthermore f^1 and f^2 are both profile vectors of k-Sperner families on $\mathbf{C}_1, \mathbf{C}_2$, the proof is completed. ■

The case of complement-free families is very analogous (and even simpler), therefore we just sketch the proof.

Theorem 15 *The extreme points of the convex hull of the profile vectors of complement-free families are $\mathbf{x}_{A,\emptyset}$ with the property that i and $n - i$ cannot be both in A, and additionally $\mathbf{x}_{A,\{n/2\}}$ if n is even.*

Proof. It is easy to see that it is enough to solve the problem reduced to a pair of maximal complement chains. There the statement holds, since there a complement-free family can contain at most two sets out of the four with size i or $n-i$, and the vectors $(1,1),(0,1),(1,0)$ are convex combinations of the vectors $(2,0),(0,2),(0,0)$. ■

1.5 Search with lies

The basic question of search theory is the following: we are given an underlying set and a family of its subsets. We can ask the members of the family in order to find a defective element. The answers show if the defective element is in the subset or not. The goal is to find the defective element.

One can easily see that the following is a special case of the previous problem. Suppose we are given distinct numbers $x_1, x_2, ..., x_n$ and at each step of our algorithm, we can ask whether $x_i < x_j$ or $x_i > x_j$ holds for any $i \neq j$. The goal is to find the order of the numbers.

There are many possible generalizations. Here we do not want to find the defective element of the underlying set, i.e. the actual order of the numbers, we only want to find out which class of a given partition contains it. The partition of the set of orderings is given by the maximum and minimum elements, i.e. we want to find those. Moreover, we allow lies, so some of the answers might turn out to be erroneous at the end.

Search problems when some of the answers may be lies have been studied by various researchers (for a list of references see the surveys by Deppe [8] and Pelc [32]). Three models have attracted the most attention. In the first model a fixed number k of the answers may be false, in the second model a fixed proportion p of the answers may be erroneous, while in the third model every answer turns out to be a lie with probability p independently from all other answers.

Here we address the problem of finding the maximum and minimum elements of an ordered set of size n using pairwise comparisons in the first model. That is, we are given distinct numbers $x_1, x_2, ..., x_n$ along with a positive integer k and at each step of our algorithm, we can ask whether $x_i < x_j$ or $x_i > x_j$ holds for any $i \neq j$, and during the process at most k answers might turn out to be false. We give an almost complete solution in the case $k = 1$. The problem of finding the maximum or the minimum element without lies was solved in [35]. Aigner in [3] considered the problem of finding both the maximum and minimum elements (which we will later also refer to as the extremal elements). He obtained asymptotically tight results for the second model, but only upper and lower bounds for the first model.

In Chapter 2 we determine the maximal size of a family satisfying certain conditions. That chapter is based on our joint work with Attila Bernáth ([5]). In Chapter 3 we present the results about profile polytopes from [20]. Chapter 4 and Chapter 5 present joint results with Balázs Patkós from [22] and [23]. In Chapter 6 we consider the problem of finding the maximum and minimum elements of an ordered set using pairwise comparisons. We suppose that some of the answers might be erroneous. This is based on our joint work with Dömötör Pálvölgyi, Balázs Patkós and Gábor Wiener ([21]). In the last chapter we summarize the results and introduce some related unsolved questions which might be examined in the future.

Chapter 2

Chain-intersecting families

Katona asked the following question:

Given a family of sets $\mathcal{F}$ in which there are no three sets A, B and C satisfying $A \subsetneq B$ and $B \cap C = \emptyset$. How many sets can such an $\mathcal{F}$ contain at most?

It was solved independently by Bernáth ([4]) and me ([19]). The following generalization is our joint work ([5]). The cases of equality were also determined there.

Problem: Given the natural numbers p and q and a family of sets $\mathcal{F}$ in which there are no sets $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_p$ and $B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq B_q$ such that $A_p \cap B_q = \emptyset$. How many sets can such an $\mathcal{F}$ contain at most?

Definition 1 *A family $\mathcal{F} \subseteq 2^{[n]}$ is called (p,q)**-chain-intersecting** if there are no sets $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_p$ and $B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq B_q$ in $\mathcal{F}$ such that $A_p \cap B_q = \emptyset$ (the tops of two chains of sizes p and q in $\mathcal{F}$ always intersect).*

Observe, that the case $p = q = 1$ of this definition gives intersecting families: we will generalize the following theorem on the maximum cardinality of intersecting families.

Theorem 16 *The largest cardinality of an intersecting family is 2^{n-1}.*

Proof. An intersecting family certainly can not contain complementing sets: so from each complementing pair we can include at most one of them. It is easy to find intersecting families achieving this bound, the family of all sets containing a common element is an example. ■

This proof is included here because complementing pair free families are used in it. We give the largest cardinality of a (p,q)-chain-intersecting family over the n element set for any values of n, p and q. We do this using a generalization of complementing pair free families.

Definition 2 *A family $\mathcal{F} \subseteq 2^{[n]}$ is called r-**complementing-chain-pair-free** if there is no chain $A_1 \subsetneq A_2 \subsetneq \dots A_r$ in $2^{[n]}$ such that all sets A_i and $\overline{A_i}$ belong to $\mathcal{F}$.*

Definition 3 *$\mathcal{F}$ is a k-**antichainpair** family if $|\mathcal{F} \cap \mathcal{K}| \leq k$ for every complement chain-pair $\mathcal{K}$.*

Observe that a (p,q)-chain-intersecting family $\mathcal{F}$ is also $(p+q-1)$-complementing-chain-pair-free: if there was a chain of length $p+q-1$ belonging to $\mathcal{F}$ together with all the complements then the smallest p members and the complements of the largest q members of this chain would give a forbidden configuration.

Also observe, that an r-complementing-chain-pair-free $\mathcal{F}$ is the union of a complement-free and an r-antichainpair family. Let $\mathcal{F}_1 = \{F \in \mathcal{F} : \overline{F} \notin \mathcal{F}\}$ and $\mathcal{F}_2 = \{F \in \mathcal{F} : \overline{F} \in \mathcal{F}|\}$. Clearly $\mathcal{F}_1$ is complement-free and $\mathcal{F}_2$ is r-antichainpair.

It is easy to see that the 1-antichainpair families are the intersecting and co-intersecting Sperner families, but in general we cannot characterize the k-antichainpair families so easily.

We will call a family $\mathcal{F}$ upwards-arranged if from $F \in \mathcal{F}$ and $|\overline{F}| > |F|$ follows $\overline{F} \in \mathcal{F}$. An upwards-arranged r-complementing-chain-pair-free family of maximal size contains all sets of size greater than $n/2$.

Let us make the following simple observation: if $A \in \mathcal{F} \subseteq 2^{[n]}$ and $\overline{A} \notin \mathcal{F}$ then the family $\mathcal{F}' = \mathcal{F} \setminus \{A\} \cup \overline{A}$ is an r-complementing-chain-pair-free family if and only if $\mathcal{F}$ was an r-complementing-chain-pair-free family. This follows simply from the fact that $\overline{A}$ cannot belong to a forbidden configuration in $\mathcal{F}'$.

We give the largest cardinality of an r-complementing-chain-pair-free family. Define the following:

Definition 4 *For a positive integer z the **upper z levels of** $2^{[n]}$ means the family of all sets of sizes $n, n-1, \dots n-z+1$ (i.e. the upper z levels in the lattice of $2^{[n]}$). The **upper $z+1/2$ levels of** $2^{[n]}$ means the upper z levels plus the sets of size $n-z$*

containing a specific element, say 1: note that this is not half of the elements on that level, unless $z = n/2$. We introduce the following notations for these families:

$$\mathcal{F}^z = \{X \subseteq [n] : |X| \geq n - z + 1\}$$

denotes the upper z levels,

$$\mathcal{F}^{z+1/2} = \{X \subseteq [n] : (|X| \geq n - z + 1) \text{ or } (|X| = n - z \text{ and } 1 \in X)\}$$

denotes the upper $z + 1/2$ levels.

These families introduced above are the optimal r-complementing-chain-pair-free families; with these notations our theorem is the following:

Theorem 17 *If $\mathcal{F}$ is an r-complementing-chain-pair-free family then $|\mathcal{F}| \leq |\mathcal{F}^{(n+r)/2}|$.*

As a consequence we immediately get the following:

Theorem 18 *The largest cardinality of a (p, q)-chain-intersecting family is equal to the cardinality of the upper $(n + p + q - 1)/2$ levels.*

Proof. Since a (p, q)-chain-intersecting family $\mathcal{F}$ is also $(p+q-1)$-complementing-chain-pair-free family it has cardinality not more than that of the upper $(n+p+q-1)/2$ levels. But this bound is achieved: the upper $(n + p + q - 1)/2$ levels form a (p, q)-chain-intersecting family. ■

As it has been mentioned, we can write any family $\mathcal{F} \subseteq 2^{[n]}$ in the form of a disjoint union $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ where $\mathcal{F}_1$ consists exactly of those members $X \in \mathcal{F}$, for which $\overline{X}$ is also in $\mathcal{F}$. Obviously $\mathcal{F}_1$ is closed under complementation. Observe that

$$\mathcal{F} \text{ is } r\text{-complementing-chain-pair-free} \iff \mathcal{F}_1 \text{ is } (r-1)\text{–Sperner}.$$

With this observation we can easily prove Theorem 17 in the case $n + r$.

Proof. In the above decomposition of the optimal r-complementing-chain-pair-free family $\mathcal{F}$ the subfamily $\mathcal{F}_2$ will obviously contain exactly one of X or $\overline{X}$ for every $X \notin \mathcal{F}_1$. So $\mathcal{F}$ is an optimal r-complementing-chain-pair-free family if and only if $\mathcal{F}_1$ is optimal among families that are $(r - 1)$-Sperner and closed under complementation. If $n + r$ is even then, according to Theorem 3, the optimal $(r - 1)$-Sperner family is closed under complementation, so this has to be $\mathcal{F}_1$; the theorem is proved.■

This observation also shows that the following theorem is a simple corollary of Theorem 17.

Theorem 19 *The maximal size of a self-complementary k-Sperner family is* $\sum_{i=(n-k+1)/2}^{(n+k-1)/2}\binom{n}{i}$ *if $n+k$ is odd and* $2\binom{n-1}{n-k-1}+\sum_{i=(n-k+2)/2}^{(n+k-2)/2}\binom{n}{i}$ *if $n+k$ is even.*

The intersection of the set of intervals with the upper z (or $z+1/2$) levels of $2^{[n]}$ will be called the **upper z (or $z+1/2$) levels on the circle**. For convenience, the notations we introduce are the following:

$$\mathcal{G}^z=(\mathcal{F}^z\cap\mathcal{I})\setminus\{[n]\},\ \mathcal{G}^{z+1/2}=(\mathcal{F}^{z+1/2}\cap\mathcal{I})\setminus\{[n]\}.$$

We simply say that the r-complementing-chain-pair-free family $\mathcal{G}$ is *optimal* if $\sum_{G\in\mathcal{G}'}\binom{n}{|G|}\le\sum_{G\in\mathcal{G}}\binom{n}{|G|}$ for any r-complementing-chain-pair-free family $\mathcal{G}'$ (optimality in this sense will only be used on the circle).

We will need the following lemmas.

Lemma 20 *If $\mathcal{G}$ is an optimal and upwards-arranged family of intervals on the circle then* $m:=\min\{|G|:G\in\mathcal{G}\}\ge(n-r+1)/2$.

Proof. Suppose indirectly that $m<(n-r+1)/2$. For an m element set $G=[a,b]$ in $\mathcal{G}$ consider the following sequence of intervals:

$$\mathcal{S}_G=[a,b],\overline{[a,b]},[a,b+1],\overline{[a,b+1]},\dots[a,b+r-1],\overline{[a,b+r-1]}.$$

Observe that $\mathcal{S}_G\not\subseteq\mathcal{G}$, because $\mathcal{G}$ is an r-complementing-chain-pair-free family.

Denote the first member of $\mathcal{S}_G\setminus\mathcal{G}$ by A_G. We claim that $A_{G_1}\ne A_{G_2}$ for different m element sets $G_1=[a_1,b_1]$ and $G_2=[a_2,b_2]$. This is true since a set in $\mathcal{S}_{G_1}\cap\mathcal{S}_{G_2}$ can only be of the form $[a_1,b_1+k]=\overline{[a_2,b_2+l]}$ (with possibly exchanging G_1 and G_2) with $a_1=b_2+l+1$ and $a_2=b_1+k+1$, thus $l=n-k-2m$. But this set can not be equal to both A_{G_1} and A_{G_2}, since this would mean that the first $2k$ members of $\mathcal{S}_{G_1}$ and the first $2l+1$ members of $\mathcal{S}_{G_2}$ all belong to $\mathcal{G}$, but these together give a chain of length $k+l=n-2m>r-1$ belonging to $\mathcal{G}$ with all complements, contradicting the r-complementing-chain-pair-free property.

We want to do the following operation:

Exchange every m element set G in $\mathcal{G}$ by A_G.

Then we want to prove that the family obtained is again r-complementing-chain-pair-free which contradicts the optimality of $\mathcal{G}$. The proof becomes technically a little bit simpler if we do first the following.

For all m element sets $G = [a, b]$ which have $\overline{[a, b + l]}$ as A_G for some $l \geq 0$, exchange $\overline{A_G}$ in $\mathcal{G}$ by A_G: we obtain another optimal family (denoted again by $\mathcal{G}$) which might not be upwards-arranged, but all m element sets $G = [a, b]$ will have A_G in the form $[a, b + k]$ for some $k > 0$.

We do not get any new m element sets, since the only m element set in $\mathcal{S}_G$ was G, for $m < (n - r + 1)/2$, and every m element set remains in $\mathcal{G}$ (because $\mathcal{G}$ was upwards-arranged).

Let us introduce the following notation: for an m element set $G = [a, b]$ in $\mathcal{G}$ consider the following chain:

$$\mathcal{L}_G = [a, b], [a, b + 1], [a, b + 2], \ldots, [a, b + r - 1].$$

So A_G is the smallest member of $\mathcal{L}_G \setminus \mathcal{G}$. We will do the following operation: for every m element set G in $\mathcal{G}$ we substitute G by A_G.

The family obtained will be denoted by $\mathcal{G}'$. Note that $|\mathcal{G}| = |\mathcal{G}'|$, $m' = \min\{|G| : G \in \mathcal{G}'\} > m$ and

$$\sum_{G \in \mathcal{G}} \binom{n}{|G|} < \sum_{G \in \mathcal{G}'} \binom{n}{|G|} \tag{2.1}$$

since $\binom{n}{|G|} < \binom{n}{|A_G|}$ for every m element set $G \in \mathcal{G}$.

Now we will prove that $\mathcal{G}'$ is also an r-complementing-chain-pair-free family which gives the contradiction. Suppose the contrary and consider a chain $\mathcal{L} = \{A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_r\}$ such that $\mathcal{L} \cup \mathrm{co}(\mathcal{L}) \subseteq \mathcal{G}'$. Of course $\mathcal{L} \cup \mathrm{co}(\mathcal{L})$ cannot be contained in $\mathcal{G}$: suppose $\mathcal{L} \nsubseteq \mathcal{G}$ (otherwise exchange $\mathcal{L}$ and $\mathrm{co}(\mathcal{L})$). For every $A \in \mathcal{L} \cup \mathrm{co}(\mathcal{L}) \setminus \mathcal{G}$ there was an m element set $G \in \mathcal{G}$ that got substituted by A (that is $A = A_G$).

We show in two steps that there was a chain of length r that belonged to $\mathcal{G}$ together with the complements, which is a contradiction.

The first step is the following: let $k = \max\{i : A_i \notin \mathcal{G}\}$.

Let $l = |A_k|$. Observe that $k \leq l - m$. Then there was an m element set G in $\mathcal{G}$ that got substituted with A_k which means that A_k is the first member of $\mathcal{L}_G$ that was not in $\mathcal{G}$. So there are $l - m \geq k$ members of $\mathcal{L}_G$ belonging to $\mathcal{G}$ along with their complements before A_k. Then we can substitute the first k elements of $\mathcal{L}$ by the first k members of $\mathcal{L}_G$ and obtain a chain $\mathcal{L}'$.

If $\mathrm{co}(\mathcal{L}')$ still contains members of $\mathcal{G}' \setminus \mathcal{G}$ then a second step is needed; otherwise we are done. In this second step substitute $\mathcal{L}'$ by $\mathrm{co}(\mathcal{L}')$ and repeat the preceding procedure: find the largest member $A = A_G$ of $\mathcal{L}'\setminus\mathcal{G}$ and exchange the beginning of $\mathcal{L}'$ with the beginning of $\mathcal{L}_G$ to obtain a chain $\mathcal{L}''$ of the same length with $\mathcal{L}''\cup\mathrm{co}(\mathcal{L}'')\subseteq\mathcal{G}$ which is a contradiction. ■

We note that the modification of this proof will be used in the next chapter. Let us give the skeleton of the operation we used:

$(*)$ In a set family $\mathcal{F}$ exchange every set G of minimum size by the first member of $\mathcal{S}_G\setminus\mathcal{F}$.

This operation depends on the definition of $\mathcal{S}_G$.

Lemma 21 *Suppose that we are given an r-complementing-chain-pair-free family $\mathcal{G}$ on the circle not containing $\emptyset$ and $[n]$.*

Then

$$\sum_{G\in\mathcal{G}}\binom{n}{|G|}\le\sum_{G\in\mathcal{G}^{(n+r)/2}}\binom{n}{|G|}. \tag{2.2}$$

Proof. The previous statement gives our lemma in the case $n+r$ is even. The only case to consider is when $n+r$ is odd and $m=(n-r+1)/2$ in the optimal and upwards-arranged family $\mathcal{G}$.

Note that all sets of size $> n-m=(n+r-1)/2$ have to be in $\mathcal{G}$, because their complements are not there: so only sets of size between m and $n-m$ are of interest (the 'middle r level' which is well defined here).

The lemma will be proved by induction on r. If $r=1$ then the result is obvious and well known.

If $r=2$ (so $m=(n-1)/2$) then $\mathcal{G}'=\mathcal{G}\cap(\mathcal{I}_{(n-1)/2}\cup\mathcal{I}_{(n+1)/2})$ is a 2-complementing-chain-pair-free family of maximum size among families in $\mathcal{I}_{(n-1)/2}\cup\mathcal{I}_{(n+1)/2}$ (all sets have equal weight, so maximizing the weight is the same as maximizing the size). So, for every $a\in[n]$ we have $|\mathcal{K}_a\cap\mathcal{G}|\le 3$, where

$$\mathcal{K}_a=\{[a,a+m-1],\overline{[a,a+m-1]},[a,a+m],\overline{[a,a+m]}\}.$$

So if we consider

$$\sum_{a\in[n]}\sum_{G\in\mathcal{K}_a\cap\mathcal{G}'}1=\sum_{G\in\mathcal{G}'}\sum_{a:G\in\mathcal{K}_a}1 \tag{2.3}$$

then we see that the right hand side is exactly $2|\mathcal{G}'|$ while the left hand side is at most $3n$. So we really obtained that $|\mathcal{G}'|\le 3n/2$ which implies $|\mathcal{G}'|\le\lfloor 3n/2\rfloor=$

$|\mathcal{G}^{(n+r)/2} \cap \big(\mathcal{I}_{(n-1)/2} \cup \mathcal{I}_{(n+1)/2}\big)|$ which also gives

$$\sum_{G \in \mathcal{G}} \binom{n}{|G|} \leq \sum_{G \in \mathcal{G}^{(n+r)/2}} \binom{n}{|G|}. \tag{2.4}$$

Note that this method gives a tight bound only if $n+r$ is even. Here the bound is not tight, but fortunately the difference is less than one and all the numbers are integers.

Assume that $r \geq 3$. We state that all (inclusionwise) minimal members of $\mathcal{G}$ are of size m or $m+1$. Suppose indirectly that there is a minimal member $G = [a, a+m-1+l]$ of $\mathcal{G}$ with $l \geq 2$. This means that neither of the $m+1$ element sets $[a, a+m]$ and $[a+1, a+m+1]$ is in $\mathcal{G}$. But in this case their intersection, the m element $[a+1, a+m]$ (which was not in $\mathcal{G}$) could be added to $\mathcal{G}$ without ruining its r-complementing-chain-pair-free property. This is because a chain of length r in $\mathcal{G} \cup \{[a+1, a+m]\}$ has to contain intervals from every level in the middle (if the complements are also in $\mathcal{G}$), but a chain in $\mathcal{G} \cup \{[a+1, a+m]\}$ starting at $[a+1, a+m]$ will not contain any $(m+1)$-element interval. So we proved that all minimal sets in $\mathcal{G}$ are of size m or $m+1$.

If we leave out the minimal sets from $\mathcal{G}$ then we get an $(r-2)$-complementing-chain-pair-free family $\mathcal{G}^1$ (which is again upwards-arranged if $r > 3$, as can be seen easily). This is proved by the following argument: suppose there is a chain $\mathcal{L}$ of length $r-2$ such that $\mathcal{L} \cup \mathrm{co}(\mathcal{L}) \subseteq \mathcal{G}^1$. But then there had to be members of $\mathcal{G}$ contained in the minimal members of $\mathcal{L}$ and $\mathrm{co}(\mathcal{L})$: the complements of these had to be in $\mathcal{G}$, too, because of the upwards-arranged property if $m+1 < n/2$ which is true for $r > 3$ (in case $r = 3$ the chains $\mathcal{L}$ and $\mathrm{co}(\mathcal{L})$ are of length 1: they have to be a complement set pair of size $n/2 = m+1$ so the sets below them are of size m and have complement in $\mathcal{G}$, too). So there was a chain of length r in $\mathcal{G}$ with all complements in it, a contradiction.

By induction $\sum_{G \in \mathcal{G}^1} \binom{n}{|G|} \leq \sum_{G \in \mathcal{G}^{(n+r-2)/2}} \binom{n}{|G|}$. The difference $\mathcal{A} = \mathcal{G} \setminus \mathcal{G}^1$ is a Sperner family with all sets of size m or $m+1$. We use the following observation (originally due to Zoltán Füredi): a Sperner family on the circle has at most n members and if it has n members then all its elements are of equal size. If all members of $\mathcal{A}$ are of size m then $\sum_{G \in \mathcal{A}} \binom{n}{|G|} = n\binom{n}{m}$. If less than m members of $\mathcal{A}$ are of size m then $\mathcal{G}$ was not optimal, since $\mathcal{G}^{(n+r)/2}$ is strictly better. So if there are sets of size $m+1$ in $\mathcal{A}$ as well then $\sum_{G \in \mathcal{A}} \binom{n}{|G|} \leq m\binom{n}{m} + (n-m-1)\binom{n}{m+1}$.

It is easy to prove that $n\binom{n}{m} \leq m\binom{n}{m} + (n-m-1)\binom{n}{m+1}$ if $m \leq n/2-1$ which is true for $r \geq 3$. So

$$\sum_{G\in\mathcal{G}} \binom{n}{|G|} = \sum_{G\in\mathcal{G}^1} \binom{n}{|G|} + \sum_{G\in\mathcal{A}} \binom{n}{|G|} \leq$$
$$\sum_{G\in\mathcal{G}^{(n+r-2)/2}} \binom{n}{|G|} + m\binom{n}{m} + (n-m-1)\binom{n}{m+1} = \tag{2.5}$$
$$\sum_{G\in\mathcal{G}^{(n+r)/2}} \binom{n}{|G|}. \blacksquare$$

Now we can prove Theorem 17.

Proof of Theorem 17. We prove by induction on r: the case $r = 1$ is trivial. Suppose $n > r > 1$ (the case $r = n$ is again simple).

If $\mathcal{F}$ is an optimal r-complementing-chain-pair-free family then it contains at least one of the sets $\emptyset$ and $[n]$: suppose $[n] \in \mathcal{F}$.

If $\emptyset$ is also in $\mathcal{F}$ then $\mathcal{F}' = \mathcal{F} \setminus \{\emptyset\}$ is an $(r-1)$-complementing-chain-pair-free: if it contained a chain $\mathcal{L}'$ of length $r-1$ with $\mathcal{L}' \cup \mathrm{co}(\mathcal{L}') \subseteq \mathcal{F}'$ then $\mathcal{L} = \mathcal{L}' \cup \{[n]\}$ would give $\mathcal{L} \cup \mathrm{co}(\mathcal{L}) \subseteq \mathcal{F}$, a contradiction. So in this case $\mathcal{F}$ could not be optimal, since

$$|\mathcal{F}| \leq |\mathcal{F}^{(n+r-1)/2}| + 1 < |\mathcal{F}^{(n+r)/2}|. \tag{2.6}$$

We can suppose that $\mathcal{F}$ is upwards-arranged. Hence $[n] \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$. Consider the family $\mathcal{F}' = \mathcal{F}\setminus\{[n]\}$. By the cycle method the inequality $|\mathcal{F}| \leq |\mathcal{F}^{(n+r)/2}|$ follows.$\blacksquare$

Chapter 3

Profile polytopes

3.1 Antichainpair families

Antichainpair families were defined in Chapter 2 (Definition 3). The profile polytope of the class of the 1-antichainpair (intersecting and co-intersecting Sperner) families has been determined by K. Engel and P.L. Erdős([10]).

Theorem 22 *The extreme points of the profile-polytope of intersecting and co-intersecting families are the vectors* $\mathbf{x}_{A,B}$ *where* $A = \emptyset$, $|B| \leq 1$ *and* $0, n \notin B$.

Our Theorem 23 is a generalization of this result, and it will help us to determine the extreme points of the profile polytope of some other classes of families.

Theorem 23 *The extreme points of the profile-polytope of* k*-antichainpair families are the vectors* $\underline{x}_{A,B}$ *where* $2|A| + |B| \leq k$ *and* $0, n \notin A$, $|B \setminus \{0, n\}| \leq 1$.

We will need the following lemma:

Lemma 24 *The set of essential extreme points of the profile-polytope of* k*-antichainpair families on the circle is the set* Γ_k *of vectors* $\mathbf{u}_{A,B}$ *where* $2|A| + |B| = k$ *and* $0, n \notin A$, $|B \setminus \{0, n\}| \leq 1$.

The special case $k = 1$ of this lemma was used by K.Engel and P.L. Erdős to prove Theorem 22. Before the proof of this lemma some other lemmas are needed:

Lemma 25 *Let $\mathcal{G} \subseteq \mathcal{H}$ be a family on the circle, such that $\emptyset, [n] \notin \mathcal{G}$ and $|\mathcal{G}| \leq in$. Then $\mathbf{p}(\mathcal{G}) \in \mathrm{conv}(\Gamma)$, where $\Gamma = \{\mathbf{u}_{A,\emptyset} : \ |A| \leq i\}$.*

Proof. Clearly this class of families is hereditary, and if we change some coordinates of an $\mathbf{u}_{A,\emptyset} \in \Gamma$ to 0, the vector remains in Γ. So it is enough to prove that the essential extreme points are in Γ (in that case all the extreme points are in Γ, and $\mathbf{p}(\mathcal{G})$ is their convex combination).

We use the following approach: A positive weight function is always maximized in at least one of the essential extreme points. It is enough to prove, that for every positive weight function w the maximum is given by a profile vector $\mathbf{u}_{A,\emptyset}$, such that $|A| \leq i$, i. e. there is a set $A \subset \{0, \ldots, n\}$ and a family $\mathcal{G}'$ such that $|A| \leq i$, $\mathbf{p}(\mathcal{G}') = \mathbf{u}_{A,\emptyset}$ and $w(\mathcal{G}) \leq w(\mathcal{G}')$.

Let $w(j_1) \geq w(j_2) \geq \cdots \geq w(j_{n-1})$ be the order of the numbers $1, \ldots n-1$ with respect to w. Then the weight of at most in intervals is maximum if they are all the j_1 element intervals, all the j_2 element intervals, and so on, while there are no more than in intervals. Clearly, this is the union of at most i complete levels, denoted by $\mathcal{G}'$. ■

Lemma 26 *Let $\mathcal{G} \subseteq \mathcal{H}$ be a family on the circle, such that $\emptyset, [n] \notin \mathcal{G}$. Let $m = \min\{|G| : G \in \mathcal{G}\}$ and $M = \max\{|G| : G \in \mathcal{G}\}$. Suppose $m \leq n - M$ and $|\mathcal{G}| \leq in + m$. Then $\mathbf{p}(\mathcal{G})$ is in the convex hull of the vectors $\mathbf{u}_{A,B}$ where $|A| \leq i$ and $0, n \notin A \cup B$, $|B| \leq 1$.*

Proof. We follow the proof of Lemma 25. There are only few differences. We order the weights only between m and M, hence the order is $w(j_1) \geq w(j_2) \geq \cdots \geq w(j_{M-m+1})$. Then the weight of at most $in + m$ intervals is maximum if they are all the j_1 element intervals, all the j_2 element intervals, and so on, while there are no more than $in + m$ intervals. It means that all the $j_1, j_2, \ldots, j_i$ intervals are in the family of maximum weight, and m intervals of size j_{i+1}.

Let $\mathcal{G}'$ be the family of all $j_1, \ldots, j_i$ element intervals and j_{i+1} intervals of size j_{i+1} if $j_{i+1} \leq n/2$, or $n - j_{i+1}$ intervals of size j_{i+1} if $j_{i+1} > n/2$. We assumed that $m \leq j_{i+1} \leq n - M$, hence there are at least m intervals of size j_{i+1} in $\mathcal{G}'$. It follows that $w(\mathcal{G}')$ is at least the above mentioned maximum weight, hence it is at least the weight of $\mathcal{G}$. The profile of $\mathcal{G}'$ is listed in the lemma. ■

Lemma 27 *Let $\mathcal{G}$ be a $2l+1$-antichainpair family on the circle such that $\emptyset, [n] \notin \mathcal{G}$. We suppose, that for all $l' < l$ the essential extreme points of the profile polytope of the $2l'+1$-antichainpair families are the vectors in $\Gamma_{2l'+1}$. Suppose that $\mathcal{G} = \mathcal{G}^1 \cup \mathcal{G}^2$ such that $\mathcal{G}^1 \cap \mathcal{G}^2 = \emptyset$, $\mathcal{G}^1$ is $2l+1-2i$-antichainpair, where i is a positive integer, $|\mathcal{G}^2| \leq in$ and there are no $G_1 \in \mathcal{G}^1$ and $G_2 \in \mathcal{G}^2$ such that $|G_1| = |G_2|$. Then the profile of $\mathcal{G}$ is dominated by Γ_{2l+1}.*

Proof. We use induction on l. The case $l = 0$ was proved by Konrad Engel and Péter Erdős ([10]), as we mentioned. By the induction $\mathbf{p}(\mathcal{G}^1)$ is dominated by $\Gamma_{2l+1-2i}$. By Lemma 25 $\mathbf{p}(\mathcal{G}^2)$ is dominated by the vectors $\mathbf{u}_{C,\emptyset}$, where $|C| \leq i$.

Clearly for every possible $\mathbf{u}_{A,B} \in \Gamma_{2l+1-2i}$ and every possible C the following is true: $(A \cup B) \cap C = \emptyset$, hence $\mathbf{u}_{A,B} + \mathbf{u}_{C,\emptyset}$ is in Γ_{2l+1}. It is easy to see that the sum of $\mathbf{p}(\mathcal{G}^1)$ and $\mathbf{p}(\mathcal{G}^2)$ (which is $\mathbf{p}(\mathcal{G})$) is a convex combination of the sum of the $\mathbf{u}_{A,B}$s and $\mathbf{u}_{C,\emptyset}$s, and these sums are all in Γ_{2l+1}. ■

Proof of Lemma 24. If $\mathbf{u}_{A,B} \in \Gamma_k$, then the family $\mathcal{G}_{A,B}$ (defined in Section 1) shows that $\mathbf{u}_{A,B}$ is a profile vector.

We use induction on k. As it was mentioned before, the case $k = 1$ is known, the case $k = 0$ is trivial. If $\emptyset$ and/or $[n]$ are in the family, the other sets form a $(k-2)$- or $(k-1)$-antichainpair family, so by induction it is enough to prove the statement for the reduced profiles.

Let $\mathcal{A}_i = \{[x, i] : x \in [n], x \neq i+1\}$ be the family of intervals, which "end" in i, and $\mathcal{B}_i = \{[i, y] : y \in [n], y \neq i-1\}$. Then $\mathcal{A}_i \cup \mathcal{B}_{i+1}$ is a chain-pair for every i. Thus $|\mathcal{G} \cap (\mathcal{A}_i \cup \mathcal{B}_{i+1})| \leq k$. If we count the elements of $\mathcal{G}$ in all $\mathcal{A}_i$s and $\mathcal{B}_j$s, we count every interval twice. On the other hand we get at most kn. Hence in the case $k = 2l$ $|\mathcal{G}| \leq nl$, and Lemma 25 finishes the proof.

Suppose that $k = 2l+1$. The previous computation shows that $|\mathcal{G}| \leq nl + n/2$.

Let $\mathcal{G}$ be a family, which does not contain $\emptyset$ and $[n]$. If there is a complete level (all i element intervals), which is a subfamily of $\mathcal{G}$, let $\mathcal{G}^2$ be this level, and $\mathcal{G}^1$ be the family of the other sets in $\mathcal{G}$. By Lemma 27 we are done.

So we can assume that there is no complete level in $\mathcal{G}$. Suppose indirectly that $\mathbf{p}(\mathcal{G})$ is not in $\mathrm{conv}(\Gamma)$. Let $m = \min\{|G| : G \in \mathcal{G}\}$ and $M = \max\{|G| : G \in \mathcal{G}\}$. We can assume that $m \leq n-M$ (otherwise we can exchange all elements by their complements, then we get a convex combination and there we can exchange every coordinate i to coordinate $n-i$). Then we can also assume by Lemma 26 that $|\mathcal{G}| > nl + m$.

If $m < \frac{n-l}{2}$, then we change $\mathcal{G}$. The skeleton of the algorithm is the following: we exchange the intervals of minimum size by either their complements or their shade. If some new intervals are already in the family, we repeat this procedure using them instead of the intervals of minimum size, i.e. we exchange them by either their complements or their shade. This algorithm ends in at most k steps. As we will se, after that the family is still k-antichainpair and the minimum size of intervals has increased.

Let $\mathcal{G}_0 = \mathcal{G}$, let m_i and M_i be the size of minimal, resp. the maximal elements of $\mathcal{G}_i$. We get $\mathcal{G}_{i+1}$ by the following steps:

Step 0a. If $m_i > n - M_i$, then exchange all M_i element sets by their complements. After that $m_i \leq n - M_i$. If there are some $n - m_i$ element sets, we have a Step 0b:

Step 0b. If there is a pair $G, \overline{G}$ such that $|G| = m_i$, then delete all the m_i-element sets whose complements are not in $\mathcal{G}_i$, and put in their complements, we denote the new family by $\mathcal{G}_i^1$. Let $\mathcal{D}_1 = \{G \in \mathcal{G} : |G| = m_i \text{ and } \overline{G} \in \mathcal{G}\}$. Otherwise exchange all $n - m_i$-element sets by their complements, let $\mathcal{D}_1$ be the family of m_i-element sets and $\mathcal{G}_i^1 = \mathcal{G}_i$.

Step 1. Let $\mathcal{G}_i^2 = \mathcal{G}_i^1 \cup \nabla_{\text{int}} \mathcal{D}_1 \setminus \mathcal{D}_1$, and $\mathcal{D}_2 = \mathcal{G}_i^1 \cap \nabla_{\text{int}} \mathcal{D}_1$.

Step 2j. Let $\mathcal{G}_i^{2j+1} = \mathcal{G}_i^{2j} \cup \text{co}(\mathcal{D}_{2j}) \setminus \mathcal{D}_{2j}$ and $\mathcal{D}_{2j+1} = \text{co}(\mathcal{G}_i^{2j}) \cap \mathcal{D}_{2j}$. We say the algorithm meets the level $n - m - j$ in this step, since the size of the new members is $n - m - j$.

Step 2j+1. Let $\mathcal{G}_i^{2j+2} = \mathcal{G}_i^{2j+1} \cup \nabla_{\text{int}} \mathcal{D}_{2j+1} \setminus \mathcal{D}_{2j+1}$ and $\mathcal{D}_{2j+2} = \mathcal{G}_i^{2j+1} \cap \nabla_{\text{int}} \mathcal{D}_{2j+1}$. We say the algorithm meets the level $m + j + 1$ in this step, since the size of the new members is $m + j + 1$.

This algorithm runs while $\mathcal{D}_j \neq \emptyset$. Let $\mathcal{G}_{i+1}^*$ be the last $\mathcal{G}_i^j$. It is easy to see, that if $\mathcal{G}_i$ is $2l' + 1$-antichainpair, then there are at most $2l'$ steps, so the cardinality of a new interval is either at most $m_i + l' - 1$ or at least $n - (m_i + l' - 1)$. An important observation: there are no m_i-element intervals in $\mathcal{G}_{i+1}^*$. In fact those are all the deleted intervals.

Last step. For every j, if there are n j-element intervals in $\mathcal{G}_{i+1}^*$, delete all of them (delete the whole levels). In this way we get $\mathcal{G}_{i+1}$.

Let the number of deleted levels before we get $\mathcal{G}_i$ be q_i. We iterate this algorithm while $m_i < \frac{n-l-2q_i}{2}$.

Claim 1. If $\mathcal{G}_i$ is $2l'+1$-antichainpair and $m_i < \frac{n-l'}{2}$, then $\mathcal{G}^*_{i+1}$ is $2l'+1$-antichainpair, too.

Proof of Claim 1. Suppose indirectly, that there are $A_1 \subset \cdots \subset A_x$ and $B_1 \subset \cdots \subset B_y$ chains in $\mathcal{G}^*_{i+1}$, such that $x+y = 2l'+1$ and there is a chain $\mathcal{C} = \{C_0, \ldots, C_n\}$, such that every A_j and every $\overline{B}_j$ is an element of $\mathcal{C}$. For example A_z is not in $\mathcal{G}_i$. Let the size of A_z be $m_i + l^*$, and $|B_{w-1}| < n - m_i - l^* \leq |B_w|$.

Our algorithm gives A_z. If it is in the Step 2j+1, there are intervals $D_{m_i} \subset D_{m_i+1} \subset, \ldots, \subset D_{m_i+l^*-1}$ in $\mathcal{G}_i$, such that their complements are in $\mathcal{G}_i$ too, $|D_t| = t$ for every t and $D_{m_i+l^*-1} \subset A_z$. (In this case $l^* = j$.) Clearly, $D_{m_i}, \ldots, D_{m_i+l^*-1}, C_{m_i+l^*}, \ldots, C_n$ form a part of a chain $\mathcal{C}'$. Therefore

$$\mathcal{G}_i \cap (\mathcal{C}' \cup co(\mathcal{C}')) \supseteq \{D_{m_i}, \ldots, D_{m_i+l^*-1}, A_{z+1}, \ldots, A_x,$$

$$B_1, \ldots, B_{w-1}, \overline{D}_{m_i+l^*-1}, \ldots \overline{D}_{m_i+1}\}.$$

These are at least $2l'+1$ intervals. The cardinalities of $A_1, \ldots A_z$ are at least m_i+1 and at most $m_i + l^*$, and we exchange these intervals by $D_{m_i}, \ldots, D_{m_i+l^*-1}$, so we get l^* intervals in place of at most l^* intervals. Similarly, the cardinalities of $B_w, \ldots, B_y$ are at least $n - m_i - l^*$ and at most $n - m$ or $n - m - 1$, depending on Step 0b. We exchange these intervals by $\overline{D}_{m_i+l^*-1}, \ldots \overline{D}_{m_i+1}$ and maybe $\overline{D}_{m_i}$, depending on Step 0b. We get l^*+1 or l^* intervals in place of at most l^*+1 or l^* intervals.

Hence $\mathcal{G}_i \cap (\mathcal{C}' \cup co(\mathcal{C}'))$ is a forbidden configuration (in $\mathcal{G}_i \cup \mathcal{G}^*_{i+1}$), and there are fewer new elements in it. We repeat this procedure until there are no new elements, which is a contradiction.

If after some repeats all the remaining intervals from $\mathcal{G}^*_{i+1} \setminus \mathcal{G}_i$ were given in the Step 2j, the procedure is similar, hence we omit the details and the proof.

Sketch: We can exchange $A_z, \ldots, A_y$ and $B_1, \ldots, B_{w-1}$ by a part of a chain-pair, which is given by our algorithm. Then we get a forbidden configuration with fewer new elements. ■

Let us continue with some simple observations.

Observation 1. $\min\{|G| : G \in \mathcal{G}^*_{i+1}\} = m_i + 1$.

Observation 2. $\min\{|G| : G \in \mathcal{G}_{i+1}\} > m_i$.

Observation 3. $|\mathcal{G}^*_{i+1}| > |\mathcal{G}_i|$.

Claim 2. $|\mathcal{G}_i^{2j+2}| + |\mathcal{D}_{2j+2}| > |\mathcal{G}_i^{2j+1}| + |\mathcal{D}_{2j+1}|$.

Proof of Claim 2. By definition $\mathcal{G}_i^{2j+2} = \mathcal{G}_i^{2j+1} \cup \nabla_{\text{int}} \mathcal{D}_{2j+1} \setminus \mathcal{D}_{2j+1}$ and $\mathcal{D}_{2j+2} = \mathcal{G}_i^{2j+1} \cap \nabla_{\text{int}} \mathcal{D}_{2+1}$. Hence $|\mathcal{G}_i^{2j+2}| + |D_{2j+2}| = |\mathcal{G}_i^{2j+1}| + |\nabla_{\text{int}} \mathcal{D}_{2j+1}|$. We have to show that $|\nabla_{\text{int}} \mathcal{D}_{2j+1}| \geq |\mathcal{D}_{2j+1}|$. It is easy to see that the operator ∇_{int} increases the size of a family except the case that family is a whole level. Suppose indirectly that is the case, $\mathcal{D}_{2j+1}$ is a whole level. Then $\mathcal{D}_{2j}$ is a whole level too and $\mathcal{G}_i^{2j-1}$ contains its complement level.

A pair of complement levels can be involved twice in the algorithm. At first one of the levels is met by the algorithm in an odd step, and the complement level in the next step. Then the later one is met in an odd step and the first one in the next step.

Our assumption was both levels were full before an odd step, hence they have to became full after the first odd and even steps. One of the levels can became full after the odd step, but if the other one becomes full during the next step, the first cannot remain full, which is a contradiction. ■

Claim 3. $|\mathcal{G}_i| \geq |\mathcal{G}| + m_i - m - q_i n$.

Proof of Claim 3. We use induction on i. The case $i = 0$ is trivial. It is enough to prove, that $|\mathcal{G}_i| \geq |\mathcal{G}_j| + m_i - m_j - (q_i - q_j)n$ for a $j < i$.

If there is a number $j < i$ and an interval $G \in \mathcal{G}_i \setminus \mathcal{G}_j$, then let j be the biggest such number. G is an interval of size at least m_i, or a complement of an interval of size at least m_i, hence there are at least $2(m_i - m_j)$ steps in the $j+1$st iteration. There are at least $m_i - m_j$ odd steps, and Claim 2 shows in these steps the size is increased by at least 1. All the decreasing is $(q_i n - q_j n)$ so the change of the size between $\mathcal{G}_j$ and $\mathcal{G}_i$ is at least $m_i - m_j - (q_i n - q_j n)$, and the proof is done.

If $\mathcal{G}_i \nsubseteq \mathcal{G}$, then there is a $G \in \mathcal{G}_i$ which is not in $\mathcal{G} = \mathcal{G}_0$, hence $j = 0$ finishes the proof.

If $\mathcal{G}_i \subseteq \mathcal{G}$, then all the new intervals have been deleted, some of them as a member of a whole level, others as intervals with minimum size. $\mathcal{G}^1 := \mathcal{G}_i$ and $\mathcal{G}^2 := \mathcal{G} \setminus \mathcal{G}_i$. Clearly $|\mathcal{G}^1| \geq \mathcal{G} - q_i n$, since the size can decrease only when whole levels are deleted, and the entire decreasing is $q_i n$. Thus $|\mathcal{G}^2| \leq q_i n$. If there are no $G_1 \in \mathcal{G}^1$ and $G_2 \in \mathcal{G}^2$ such that $|G_1| = |G_2|$, we can apply Lemma 27, and we proved the Lemma 24, which is a contradiction (we supposed indirectly, that the lemma is not true). If there are $G_1 \in \mathcal{G}^1$ and $G_2 \in \mathcal{G}^2$ such that $|G_1| = |G_2| = a$, then $a \geq m_i$. All the a-element intervals were deleted at least once during the algorithm, i.e. they are members of

$\mathcal{G}_j^* \setminus \mathcal{G}_j$ for a $j < i$. Then $G_1 \in \mathcal{G}_i \setminus \mathcal{G}_j$, and this finishes the proof. ■

It is important to see, that this claim is not true in general, it follows from the indirect assumption.

Finally we get a $2l' + 1$-antichainpair, $\mathcal{G}'$, and the cardinalities of the intervals are at least $\lceil \frac{n-l'}{2} \rceil$, at most $\lceil \frac{n+l'}{2} \rceil$. There are no whole levels in $\mathcal{G}'$.

Claim 4. $|\mathcal{G}'| \le \lfloor n/2 \rfloor + l'(n-1) + \lfloor l'/2 \rfloor$.

Proof of Claim 4. Now there can be only $l'+1$ nonempty levels. Here a forbidden configuration would consist of a chain of length $l'+1$, and all the complements. Therefore if $G, \overline{G} \notin \mathcal{G}'$, we can put in one. Exchange all intervals of size less than $n/2$ by their complements, if the complement is not in $\mathcal{G}'$. After that, if $n+l'$ is odd, exchange all intervals of size $\lceil \frac{n+l'}{2} \rceil$ by their complements (which cannot be in $\mathcal{G}'$). Now we are given a $2l' + 1$-antichainpair, $\mathcal{G}''$, such that $|\mathcal{G}''| = |\mathcal{G}'|$. In $\mathcal{G}''$ there are $\lceil l'/2 \rceil$ levels of size greater than $n/2$, which are not empty, and if they are not whole, we can put in the missing intervals, because their complements are not in $\mathcal{G}''$. There are no whole levels of size less than $n/2$ in $\mathcal{G}''$ (because there are no whole levels at all in $\mathcal{G}'$).

Let $\mathcal{A}$ be the family of intervals of size $\lfloor n/2 \rfloor$, such that there is a chain of length $l' + 1$ in $\mathcal{G}''$ containing this interval. Suppose $A, A' \in \mathcal{A}$ and $A \cap A' = \emptyset$. There are chains $B_1 \subset \dots B_x \subset A$ and $C_1 \subset \dots C_x \subset A'$ in $\mathcal{G}''$. Then $B_1, \dots B_x, \dots A, \overline{C}_x \dots \overline{C}_y$ and $C_1, \dots C_x, \dots A, \overline{B}_x \dots \overline{B}_y$ constitute a forbidden configuration, where $y = 1$ if $n+l'$ is even and $y = 2$ otherwise. Hence $\mathcal{A}$ is intersecting, so $|\mathcal{A}| \le \lfloor n/2 \rfloor$.

Clearly $\mathcal{G}'' \setminus \mathcal{A}$ is an l'-Sperner family, so its size is at most $l'n$. Moreover an l'-Sperner family is the union of l' Sperner families. It is an easy exercise to see that a Sperner family can contain n intervals only if there is a j such that the family contains all j element intervals. $\mathcal{G}'' \setminus \mathcal{A}$ can contain all j element intervals only if $j > n/2$, hence at most $\lfloor l'/2 \rfloor$ times. Hence $|\mathcal{G}'' \setminus \mathcal{A}| \le l'(n-1) + \lfloor l'/2 \rfloor$, so $|\mathcal{G}'| = |\mathcal{G}''| \le \lfloor n/2 \rfloor + l'(n-1) + \lfloor l'/2 \rfloor$. ■

Suppose that $\mathcal{G}' = \mathcal{G}_i$. Then $|\mathcal{G}'| \ge |\mathcal{G}| + m_i - m - q_i n$. Clearly $q_i = l - l'$ and $m_i = \lceil \frac{n-l'}{2} \rceil$. Hence

$$|\mathcal{G}| \le |\mathcal{G}'| - \lceil \frac{n-l'}{2} \rceil + m + nl - l'n \le \lfloor n/2 \rfloor + l'(n-1) + \lfloor l'/2 \rfloor - \lceil \frac{n-l'}{2} \rceil + m + nl - l'n$$

$$\le m + nl + \lfloor n/2 \rfloor - l' + \lfloor l'/2 \rfloor - \lceil \frac{n-l'}{2} \rceil \le m + nl,$$

which is a contradiction. It finishes the proof of Lemma 24. ■

Proof of Theorem 23. One can easily see, that it is enough to prove the theorem for reduced profiles. If $\mathbf{x}_{A,B}$ is one of the listed vectors, $\mathcal{F}_{A,B}$ is k-antichainpair, hence these vectors are profile vectors. Now we can apply Lemma 24 and Theorem 8, and we are done.$\blacksquare$

3.2 Corollaries

Theorem 28 *The essential extreme points of the complement-free k-antichainpair families are $\mathbf{x}_{A,B}$ where $2|A|+|B|=k,\ 0,n/2,n\notin A$, $|B\setminus\{0,n/2,n\}|\leq 1$, and $i\in A\cup B$ implies $n-i\notin A\cup B$ except for $i=n/2$.*

Proof. It is easy to see, that these vectors $\mathbf{x}_{A,B}$ are essential extreme points. Let w be a positive weight function and $\mathcal{F}$ be the optimal family for this weight.

For any A, only one of A or $\overline{A}$ can belong to $\mathcal{F}$. If any of them belongs to $\mathcal{F}$, it is the one which has larger weight, since otherwise we could exchange it to its complement. If $w(i)<w(n-i)$ then clearly $\mathcal{F}$ does not contain i element sets, if $w(i)=w(n-i)$, where $i\neq n/2$, then we choose one of them, i, and exchange all $n-i$ element sets of $\mathcal{F}$ to its complement. It does not change the weight, and $\mathcal{F}$ remains complement-free k-antichainpair.

Let $w'(i)=0$, if $w(i)<w(n-i)$ or if $w(i)=w(n-i)$ and $i<n-i$. Otherwise let $w'(i)=w(i)$. Then the optimal complement-free k-antichainpair family for this weight will be also optimal for w. Let $\mathcal{K}_0$ be an optimal k-antichainpair for the weight w', and delete all the sets with weight 0. Then we get a family $\mathcal{K}_1$, which is also optimal for this weight, and almost complement-free: if A and $\overline{A}$ both are in $\mathcal{K}_1$, then $|A|=n/2$.

If n is odd, we are done: $\mathcal{K}_1$ is maximal for w', and one can easily see that it is maximal for w, and its profile is listed in the theorem.

If n is even, we can assume, that all the $n/2$ element sets of $\mathcal{F}$ contain 1 (we can exchange all the others to their complements without violating the complement-free k-antichainpair property or changing the weight).

Case 1. $\mathcal{K}_1$ does not contain all the $n/2$ element sets. Then it contains at most $\binom{n-1}{n/2-1}$ members of size $n/2$. Its profile vector is $\mathbf{x}_{A,B}$. We can assume this is the family $\mathcal{F}_{A,B}$. Then it is complement-free, and we are done.

Case 2. The optimal (for w') $(k+1)$-antichainpair, $\mathcal{K}_2$ contains $n/2$ element sets. Then let $\mathcal{A} = \{A \subset [n] : |A| = n/2,\ 1 \notin A\}$. We can assume that $\mathcal{K}_2$ contains $\mathcal{A}$. Let $\mathcal{K}_3 = \mathcal{K}_2 \setminus \mathcal{A}$, then it is a complement-free k-antichainpair and its profile vector is listed in the theorem. Let $\mathcal{F}' = \mathcal{F} \cup \mathcal{A}$. Then $w'(\mathcal{F}) = w'(\mathcal{F}') - \binom{n}{n/2}w'(n/2) \leq w'(\mathcal{K}_2) - \binom{n}{n/2}w'(n/2) = w'(\mathcal{K}_2 \setminus \mathcal{A})$, hence $\mathcal{K}_2 \setminus \mathcal{A}$ is (also) an optimal family, which has profile listed in the theorem.

Case 3. The optimal for w' k-antichainpair $\mathcal{K}_1$ contains all $n/2$ element sets, and the optimal $(k+1)$-antichainpair $\mathcal{K}_2$ does not contain any $n/2$ element sets. Let the profile vector of $\mathcal{K}_1$ (resp. $\mathcal{K}_2$) be $x_{A,B}$ ($x_{C,D}$).

Case 3.1. $|C| \geq |A|$. We know that $n/2 \in A \setminus C$, hence there is a $j \in C \setminus A$, where $j \neq 0, n/2, n$. If $\binom{n}{n/2}w(n/2) \leq \binom{n}{j}w(j)$, then we can exchange the $n/2$ element sets in $\mathcal{K}_1$ by the j element sets, and we come into Case 1, we found a family which is optimal for w' and does not contain all the $n/2$-element sets. If $\binom{n}{n/2}w(n/2) \geq \binom{n}{j}w(j)$, then we can exchange the j element sets in $\mathcal{K}_2$ by the $n/2$ element sets, and we come into Case 2.

Case 3.2. $|C| < |A|$. $|A| \leq \lfloor k/2 \rfloor$, hence $|C| \leq \lfloor k/2 \rfloor - 1$. On the other hand $2|C| + |D| = k+1$ and $|D| \leq 3$. It is possible only if $|A| = \lfloor k/2 \rfloor$ and $|C| = \lfloor k/2 \rfloor - 1$. Moreover $2|A| + |B| = k$, hence $|D| - |B| = 3$. It means $|D| = 3$ and $|B| = 0$. Then $\emptyset$ and $[n]$ are in $\mathcal{K}_2$ and not in $\mathcal{K}_1$. The family of $(n/2)$-element sets is not in $\mathcal{K}_2$, hence the weight w' of all the $(n/2)$-element sets is not more than the weight of $\emptyset$ and $[n]$ (otherwise we could exchange them). Then we can exchange the $(n/2)$ element sets in $\mathcal{K}_1$ by $\emptyset$ and $[n]$, and we come into Case 1. ■

Theorem 29 *The essential extreme points of the profile-polytope of the class of r-complement-chain-pair-free families are $\mathbf{x}_{A,B}$, where $2|A| + |B| = n + r$, $i \notin A$ implies $n - i \in A$, except for $i = n/2$. In addition $0, n \notin B$, $|B \setminus \{n/2\}| \leq 1$.*

Proof. We determine the reduced essential extreme points, the rest of the proof is trivial. Let w be a positive weight function and $\mathcal{F}$ be an optimal r-complement-chain-pair-free family. We can assume that $\mathcal{F}$ is maximal, i.e. if $\mathcal{F} \cup \{F\}$ is r-complement-chain-pair-free, then $F \in \mathcal{F}$. It follows easily, that if $A \notin \mathcal{F}$, then $\overline{A} \in \mathcal{F}$. So $\mathcal{F}$ contains one element of every pair of complements, of course which has greater weight, otherwise we could exchange it to its complement.

Let $\mathcal{F}_0$ be their family, if $w(i) = w(n-i)$, then choose one, for example i, and from all pairs $A, \overline{A}$ if $|A| = i$ and $A \notin \mathcal{F}$, then put A instead of $\overline{A}$ in $\mathcal{F}$ (and $\mathcal{F}_0$). Then $\mathcal{F}$ remains an optimal r-complement-chain-pair-free family, and in $\mathcal{F}_0$ there are only whole levels, except for the level $n/2$.

Let $\mathcal{F}_1 = \mathcal{F} \setminus \mathcal{F}_0$, it is a complement-free r-antichainpair. Let $w'(i) = 0$ if all the i element sets are in $\mathcal{F}_0$, and $w'(i) = w(i)$ otherwise. Clearly the weight of $\mathcal{F}_1$ does not change. Theorem 28 says, what vectors $\mathbf{x}_{A,B}$ are maximal for w'; clearly we can suppose that if $w'(i) = 0$ then $i \notin A \cup B$. It is easy to see, that $\mathbf{p}(\mathcal{F}_0) + \mathbf{x}_{A,B}$ is listed in the theorem.■

The extreme points of the profile-polytope of the k-Sperner families are determined in Theorem 12, and the extreme points of complement-free Sperner families are also known ([10], [11]). Moreover, in Theorem 13 we also found the extreme points of the complement-free k-Sperner families in the case n is odd.

Theorem 30 *The essential extreme points of the profile-polytope of the complement-free k-Sperner families are $\mathbf{x}_{A,B}$, where $2|A| + |B| = 2k$, $0, n \notin B$, $i \in A \cup B$ implies $n - i \notin A \cup B$ except for $i = n/2$, $n/2 \notin A$ and $|B \setminus \{n/2\}| \leq 1$.*

Proof. Clearly, $\mathcal{F}_{A,B}$ is complement-free k-Sperner with such profile. One can easily see, that it is enough to prove the theorem for reduced profiles. The complement-free k-Sperner families are complement-free $2k$-antichainpairs, which by Theorem 28 gives the statement. ■

The maximal size of intersecting Sperner-families was determined by Milner ([30]). It is equal to the maximal size of complement-free Sperner families, which was determined by Purdy ([34]). Here we prove the following generalization:

Corollary 31 *Let $\mathcal{F}$ be an intersecting (or a complement-free) k-Sperner family. Then*

$$\mathcal{F} \leq \begin{cases} \displaystyle\sum_{i=\frac{n+1}{2}}^{\frac{n+1}{2}+k-1} \binom{n}{i} & \text{if } n \text{ is odd} \\ \binom{n-1}{n/2-1} + \displaystyle\sum_{i=n/2+1}^{n/2+k-1} \binom{n}{i} + \binom{n-1}{n/2+k} & \text{if } n \text{ is even.} \end{cases}$$

Proof. Clearly $\mathcal{F}$ is a complement-free k-Sperner family. It is easy to see, that the statement follows applying the weight $w = 1$. ■

One can also easily see that this bound is tight.

The maximal size of self-complementary k-Sperner families was determined in [5], the extreme points of the self-complementary Sperner families are also known ([10]).

Theorem 32 *The extreme points of the profile-polytope of the self-complementary k-Sperner families are $\mathbf{x}_{A,B}$, where $2|A|+|B| \leq 2k$, $0, n/2, n \notin B$, $i \in A$ implies $n-i \in A$, $i \in B$ implies $n-i \in B$ and either $|B| = 2$ and $|A| = k-1$ or $|B| = 0$.*

Proof. Let w be a weight function and define $w'(i) = w'(n-i) = \frac{w(i)+w(n-i)}{2}$. Clearly the weight of a self-complementary family does not change.

Let Γ be the set of the vectors $\mathbf{x}_{A,B}$, where $2|A| + |B| \leq 2k$, $0, n/2, n \notin B$, $i \in A$ implies $n-i \in A$, $i \in B$ implies $n-i \in B$ and $|B| \leq 2$. Clearly these are the same properties as those in the theorem, except for the last one. Hence Γ contains more vectors, but it is easy to see that they are in the convex hull of the vectors listed in the theorem. In fact it is easier to see that for any weight w the weight of a vector in Γ cannot exceed the weight of all vectors listed in the theorem. The maximal weight in Γ is $w(\mathbf{x}_{A,B}) = w'(\mathbf{x}_{A,B}) = w'(\mathcal{F}_{A,B})$, and we can suppose $\mathcal{F}_{A,B}$ does not contain any sets of negative weight w' (it might contain some sets F such that $w(F) < 0$). If $\mathbf{x}_{A,B}$ is not listed in the theorem, than $|B| = 2$ and $|A| < k-1$. But then we could add the other sets from the levels contained in B without violating the property or decreasing the weight.

Thus it is enough to prove that all the profile vectors of self-complementary k-Sperner families are in the convex hull of Γ.

Let $\mathcal{F}$ be the maximal for w' self-complementary k-Sperner family. $\mathcal{F}$ contains pairs $F, \overline{F}$. We define a partition of $\mathcal{F}$: for all the pairs $F, \overline{F}$ if $|F| < |\overline{F}|$ or $|F| = |\overline{F}|$ and $1 \in F$ then $F \in \mathcal{F}^1$ and $\overline{F} \in \mathcal{F}^2$. $\mathcal{F}^1$ and $\mathcal{F}^2$ are complement-free k-antichainpairs, such that all members of $\mathcal{F}^1$ (resp. $\mathcal{F}^2$) are of size at most (at least) $n/2$. Let $w''(i) = w'(i)$ if $i \leq n/2$ and 0 otherwise, and $w'''(i) = w''(n-i)$. Clearly $w''(\mathcal{F}^1) = w'(\mathcal{F}^1)$ and $w'''(\mathcal{F}^2) = w'(\mathcal{F}^2)$. Theorem 28 gives the maximal for w'' complement-free k-antichainpair (here the weight can be negative, so we have to use (1) of Proposition 6). It is easy to see, that the complement of this family is the maximal for w''' complement-free k-antichainpair. The union of this complement-free k antichainpair and its complement family is a self-complementary k-Sperner, and the profile vector of it is in Γ. Its weight is at least the weight of $\mathcal{F}$. ■

Chapter 4

Chain Profile polytopes

The following definitions and results are from our joint work with Balázs Patkós ([22]). In the previous chapters the weight of a family was the sum of the weight of its members. However, there are problems dealing with other kind of weight functions, and problems not dealing with members of some families, but subfamilies of families. A natural question is the following: let $l \leq k$ be two integers, how many l-chains can be contained in a family without a $k+1$-chain (i.e. in a k-Sperner family). Note, that this problem is analogous to the celebrated theorem of Turán [39], generalized by Sauer [37] (see also [7]) stating that for all integers $2 \leq r \leq s$ a graph on n vertices without a clique of size $s+1$ can contain at most as many cliques of size r as does the s-partite Turán graph on n vertices (the complete s-partite graph with equipartite partition).

To deal with the above problem we introduce the notion of the **chain profile vector** of a family $\mathcal{F}$ on an n-element underlying set. This has 2^{n+1} components, and the αth component f_α, where $\alpha = (\alpha_1, ..., \alpha_l)$ with $0 \leq \alpha_1 < \alpha_2 < ... < \alpha_l \leq n$, denotes the number of l-chains in $\mathcal{F}$ in which the smallest set has size α_1, the second smallest has size α_2, and so on. Similarly we define the **l-chain profile vector** consisting only the components α of size l. Note that for $l = 1$ this is the usual the profile vector.

4.1 Definitions and remarks

In this section we give some further definitions and describe some basic connections between the extreme points in the l-chain case and the extreme points in the original

(1-chain) case.

Notation. For $\alpha = (\alpha_1, \alpha_2, ..., \alpha_l), 0 \leq \alpha_1 < \alpha_2 < ... < \alpha_l \leq n$ we define the following multinomial coefficient:

$$\binom{n}{\alpha} = \prod_{i=1}^{l} \binom{n - \alpha_{i-1}}{\alpha_i - \alpha_{i-1}} = \frac{n!}{\alpha_1!(\alpha_2 - \alpha_1)!...(\alpha_l - \alpha_{l-1})!(n - \alpha_l)!}$$

where $\alpha_0 = 0$ and $0! = 1$ as usual. Note that $\binom{n}{\alpha}$ is the number of l-chains that can be formed from subsets of an n-element set in such a way that the smallest set has size α_1, the second smallest has size α_2 and so on.

Definition 5 *Let $\mu_l(\mathbf{A})$ denote the set of all l-chain profile vectors of families in $\mathbf{A}$, $\langle\mu_l(\mathbf{A})\rangle$ its convex hull, $\mathcal{E}_l(\mathbf{A})$ the extreme points of $\langle\mu_l(\mathbf{A})\rangle$ and $E_l(\mathbf{A})$ the families from $\mathbf{A}$ with l-chain profile in $\mathcal{E}_l(\mathbf{A})$. Let furthermore $\mathcal{E}_l^*(\mathbf{A})$ denote the essential extreme points and $E_l^*(\mathbf{A})$ the corresponding families.*

Theorem 33 *For any upward or downward closed class of families $\mathbf{A} \subseteq 2^{2^X}$ and for any $l \geq 1$*

$$\mathcal{E}_l^*(\mathbf{A}) \subseteq \mu_l(E_1^*(\mathbf{A})).$$

Note that equality does not always hold as the class of intersecting families, the family $\mathcal{F} = \{F \subseteq X : |F| > |X|/2\}$ and any $l > |X|/2$ shows.

Proof. The proof is the same for downward and upward closed classes of families, so we assume that $\mathbf{A}$ is upward closed.

Let $E_1^*(\mathbf{A}) = \{\mathcal{F}_1, \mathcal{F}_2, .., \mathcal{F}_m\}$ and let f^i the profile of $\mathcal{F}_i$, $f^{i,l}$ the l-chain profile of $\mathcal{F}^i$ and $f^{i,l}_\alpha$ its αth component.

We have to prove that the l-chain profile f^l of any family $\mathcal{F}$ in $\mathbf{A}$ can be dominated by a convex combination of the $f^{i,l}$s. Denote the profile of $\mathcal{F}$ by f. Clearly we have

$$f^l_\alpha \leq f_{\alpha_1} \binom{n - \alpha_1}{\alpha^*},$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_l)$ and $\alpha^* = (\alpha_2 - \alpha_1, \alpha_3 - \alpha_1, ..., \alpha_l - \alpha_1)$. Inequality holds with equality for the f^i_αs and the $f^{i,l}_\alpha$s. The fact that the f^is are the essential extreme points of $\langle\mu_1(\mathbf{A})\rangle$ means that for some convex combination $c_i, i = 1, ..., m$

$$f \leq \sum_{i=1}^{m} c_i f^i.$$

But then

$$f^l_\alpha \le f_{\alpha_1}\binom{n-\alpha_1}{\alpha^*} \le \binom{n-\alpha_1}{\alpha^*}\sum_{i=1}^{m} c_i f^i_{\alpha_1} = \sum_{i=1}^{m} c_i f^{i,l}_\alpha,$$

which completes the proof. ■

Since the convex hull of the profile polytope of the class of intersecting families were determined by P.L. Erdős, P. Frankl and G.O.H. Katona in [15], Theorem 33 provides the essential extreme points of the convex hull of the l-chain profile polytopes.

Theorem 34 *For any convex closed set of families* $\mathbf{A} \subseteq 2^{2^X}$ *and for any* $l \ge 2$

$$\mathcal{E}^*_l(\mathbf{A}) \subseteq \mu_l(E^*_2(\mathbf{A})).$$

Proof. The proof is analogous to that of Theorem 33, the inequality needed is

$$f^l_\alpha \le f^2_{\alpha_1,\alpha_l}\binom{\alpha_l-\alpha_1}{\alpha^*}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_l)$, $\alpha^* = (\alpha_2 - \alpha_1, \alpha_3 - \alpha_1, ..., \alpha_{l-1} - \alpha_1)$ and for families with essential extreme profile inequality holds with equality. ■

Unfortunately the extreme points of neither the 1-chain, nor the 2-chain profile polytope are known for the class of intersecting and co-intersecting families.

4.2 The reduction method

In this section we describe our main tool in determining the l-chain profile polytope of families of sets with some given property. It is a modification of the permutation method.

Definition 6 *For any l let $T^l_{\mathbf{C}}$ denote the following operator acting on the $\binom{n+1}{l}$-dimensional $\mathbf{R}$-space whose coordinates are indexed by l-tuples of integers $(\alpha_1, \alpha_2, \ldots, \alpha_l)$, where $0 \le \alpha_1 < \alpha_2 < \ldots < \alpha_l \le n$.*

$$T^l_{\mathbf{C}} : e \mapsto T^l_{\mathbf{C}}(e) \qquad \text{where} \qquad T^l_{\mathbf{C}}(e)_\alpha = \binom{n}{\alpha} e_\alpha.$$

Definition 7 *For a family $\mathcal{F}$ on a base set X and a maximal chain $\mathbf{C}$ in X let $\mathcal{F}(\mathbf{C}) = \{F \in \mathcal{F} \cap \mathbf{C}\}$ and for a set of families $\mathbf{A}$ let $\mathbf{A}(\mathbf{C}) = \{\mathcal{F}(\mathbf{C}) : \mathcal{F} \in \mathbf{A}\}$.*

Theorem 35 *For any set of families* $\mathbf{A} \subseteq 2^{2^X}$ *if the extreme points* $e_1, e_2, ..., e_m$ *of* $\langle \mu_l(\mathbf{A}(\mathbf{C}))\rangle$ *do not depend on the choice of* $\mathbf{C}$*, then*

$$\langle \mu_l(\mathbf{A})\rangle \subseteq \langle \{T^l_{\mathbf{C}}(e_1), ..., T^l_{\mathcal{C}}(e_m)\}\rangle.$$

Proof. The modification of the argument in [15] works. Let $\mathcal{F}$ be an element of $\mathbf{A}$ with l-profile $f = (..., f_\alpha, ...)$. For $\mathbf{F} = \{F_1 \subset F_2 \subset ... \subset F_l\}$ with $|F_i| = \alpha_i, i = 1, ..., l$ let $\underline{w}(\mathbf{F})$ be the vector of length $\binom{n+1}{l}$ with $1/n!$ in the αth component and 0 everywhere else (where n is the size of the base set). Consider the sum $\sum \underline{w}(\mathbf{F})$ for all pairs $(\mathbf{C}, \mathbf{F})$, where $\mathbf{C}$ is a maximal chain on X and $\mathbf{F} \subset \mathcal{F} \cap \mathbf{C}$ an l-chain. For a fixed $\mathbf{C}$ we have

$$\sum_{\mathbf{F}\in\mathcal{F}(\mathbf{C})} \underline{w}(\mathbf{F}) = \frac{1}{n!}(\text{profile of } \mathcal{F}(\mathbf{C})).$$

Here the profile of $\mathcal{F}(\mathbf{C})$ is a convex linear combination $\sum_{i=1}^m \lambda_i(\mathbf{C})e_i$ of the e_is. Therefore

$$\sum_{\mathbf{C},\mathbf{F}} \underline{w}(\mathbf{F}) = \sum_{\mathbf{C}}\sum_{\mathbf{F}} \underline{w}(\mathbf{F}) = \sum_{\mathbf{C}} \frac{1}{n!}\sum_{i=1}^m \lambda_i(\mathbf{C})e_i = \sum_{i=1}^m \frac{1}{n!}(\sum_{\mathbf{C}} \lambda_i(\mathbf{C}))e_i \tag{4.1}$$

holds where $\sum_{\mathbf{C}} \frac{1}{n!}\sum_{i=1}^m \lambda_i(\mathbf{C}) = 1$. Thus $\sum \underline{w}(\mathbf{F})$ is a convex linear combination of the e_is.

Summing in the other way around, we have

$$\sum_{\mathbf{C},\mathbf{F}} \underline{w}(\mathbf{F}) = \sum_{\mathbf{F}}\sum_{\mathbf{C}} \underline{w}(\mathcal{F}) =$$

$$\sum_{\mathbf{F}}(0, 0, ..., \frac{|F_1|!(|F_2| - |F_1|)!...(|F_l| - |F_{l-1}|)!(n - |F_l|)!}{n!}, ..., 0) = (..., \frac{f_\alpha}{\binom{n}{\alpha}}, ...), \tag{4.2}$$

since for a fixed $\mathbf{F} = \{F_1 \subset F_2 \subset ... \subset F_l\}$ there are exactly $|F_1|!(|F_2| - |F_1|)!...(|F_l| - |F_{l-1}|)!(n - |F_l|)!$ chains containing $\mathbf{F}$. So (1) and (2) give that this last vector is a convex linear combination of the e_is, which implies that f is the linear combination of $T^l_{\mathbf{C}}(e_1), ..., T^l_{\mathbf{C}}(e_m)$. ∎

The structure of maximal chains is too simple, so using only them is not enough to determine the l-chain profile polytope of more difficult sets of families. But the proof of Theorem 35 works if we replace the chain by the chain-pair. In the proof one has to write (instead of $\frac{1}{n!}$) $\frac{2}{(n!)}$ in the definition of $\underline{w}(\mathbf{F})$, and modify the definition of the T-operator to

$$(T^l_{\mathbf{C}_1,\mathbf{C}_2}(e))_\alpha = \frac{1}{d_\alpha}\binom{n}{\alpha},$$

where d_α is the number of α-type l-chains in the pair of complementing chains. Actually it works analogously for any family $\mathcal{H}$ instead of the chain, but it seems to be hopeless to determine the l-chain profile polytope in a difficult case even on the circle.

4.3 Reduction to the chain

All the proofs from Section 1.4.1 work here.

Theorem 36 *For all $l \geq 1$ the extreme points of the convex hull of the l-chain profile vectors of convex families are the following:*

the all zero vector

$$\mathbf{0} = (0, ..., 0)$$

and for all $0 \leq i \leq j \leq n$ the vectors $v_{i,j}$

$$(v_{i,j})_\alpha = \begin{cases} \binom{n}{\alpha} \text{ if } & i \leq \alpha_1 < \alpha_l \leq j \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

Proof. The vector $v_{i,j}$ is the l-profile of the family $\mathcal{F}_{i,j} = \{F \subseteq [n] : i \leq |F| \leq j\}$, which is convex.

On a chain any convex family must consist of some consecutive subsets of the chain. The theorem follows now from Theorem 35. ■

Theorem 37 *For any $l \leq k$ the extreme points of the l-chain profile polytope of k-Sperner families are the following:*

the all zero vector

$$\mathbf{0} = (0, ..., 0, ...0)$$

and for all $l \leq z \leq k$ and $\beta = \{\beta_1, ..., \beta_z\}$ with $0 \leq \beta_1 < ... < \beta_z \leq n$ the vectors v_β

$$(v_\beta)_\alpha = \begin{cases} \binom{n}{\alpha} \text{ if } & \alpha \subseteq \beta \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

Proof. It is trivial to see that these vectors are l-chain profiles of the corresponding levels, and they are convex linearly independent.

A k-Sperner family on a maximal chain consists of at most k sets, therefore its l-chain profile vector have ones in those components $\alpha = (\alpha_1, ..., \alpha_l)$ for which there is an element in the family with size α_i for all $i = 1, ..., l$. All these vectors are convex independent. Therefore they form the convex hull of the profile polytope on the chain, and Theorem 35 implies now Theorem 37. ■

Applying Theorem 37 for the constant 1 weight function one gets

Corollary 38 *For any $l \leq k$ if a family $\mathcal{F}$ on an n-element base set X does not contain a chain of length $k+1$, then the number of l-chains in $\mathcal{F}$ is at most*

$$\max_{\beta \subset [0,n]; |\beta| = k} \sum_{\alpha \subseteq \beta; |\alpha| = l} \binom{n}{\alpha}. \tag{4.5}$$

If $l = k$, then this gives the maximum number of k-chains that a k-Sperner family can contain. This is $\binom{n}{\alpha}$ where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k)$ and the numbers $\alpha_1, \alpha_2 - \alpha_1, ...\alpha_k - \alpha_{k-1}$ differ by at most one. If $k+1$ divides n, then we get the uniqueness of the extremal system (take all $F \subseteq X$ with $|F| = \alpha_i$ for some $i = 1, ..., k$) automatically. If $k+1$ does not divide n, then we can lift up (4.5) to an AZ-type identity (for more details see [31], and for the original AZ-identity see the paper of Ahlswede and Zhang [2]) which will assure the uniqueness.

Theorem 37 implies (if $\mathbf{S}_k$ denotes the class of k-Sperner families) $E_1(\mathbf{S}_k) = E_l(\mathbf{S}_k)$. But the bordering faces of the convex hulls $\langle \mu_1(\mathbf{S}_k) \rangle$ and $\langle \mu_l(\mathbf{S}_k) \rangle$ are not analogous. If $l = 1$ the convex hull determined by the faces given by the inequalities $0 \leq f_i \leq \binom{n}{i}$ and the LYM-inequality (Theorem 2), while if $l > 1$ the hyperplanes given by $0 \leq f_\alpha \leq \binom{n}{\alpha}$ are bordering faces along with the LYM-type inequality

$$\sum_{\alpha} \frac{f_\alpha}{\binom{n}{\alpha}} \leq \binom{k}{l}, \tag{4.6}$$

but there are some additional ones, which can be seen by the following observation. Choosing $\binom{k}{l}$ αs in such a way that their union has size strictly larger than k and putting $f_\alpha = \binom{n}{\alpha}$ for these αs and 0 for the others, we obtain an essential extreme point of the polytope determined by the above inequalities, and which is not an l-chain profile of any k-Sperner families.

Theorem 33 (and 34) states that for a certain class of sets of families all candidates for the families with essential extreme l-chain profiles are among the families with essential extreme 1-chain (2-chain) profile. Theorem 37 states, that for k-Sperner families the above statement is true for all extreme profiles (not only for essential extreme profiles). It seems natural to conjecture that for all set of families $\mathbf{A}$ and $l > 1$ $E_l(\mathbf{A}) \subseteq E_1(\mathbf{A})$ and/or $E_l^*(\mathbf{A}) \subseteq E_1^*(\mathbf{A})$. But this is false.

The counterexample is based on Theorem 15. Note that the families corresponding to the extreme points cannot contain sets of size i and $n-i$ at the same time. Hence all 2-chain profiles of those families have 0 in their components indexed with the sets $\{i, n-i\}$, and therefore all their convex combinations have 0 in those components. But a pair of subsets in inclusion with size i and $n-i$ is of course a complement-free family, and its profile is not in the convex hull of the above-mentioned vectors.

Chapter 5

Profile vectors in the poset of subspaces

This chapter is based on our joint work with Balázs Patkós ([23]). The notion of profile vectors can be introduced for any ranked poset P. In this case the profile of a family $\mathcal{F} \subseteq P$ is defined by

$$f(\mathcal{F})_i = |\{p \in \mathcal{F} : r(p) = i\}| \quad (i = 0, 1, ..., n),$$

where n is the largest rank in P. Several results are known about profile vectors in the generalized context as well (see e.g. [9], [16], [17], [36]).

One of the most studied ranked poset is $L_n(q)$ of subspaces of a finite vector space. In this case the rank of a subspace is just its dimension, so the profile vector $f(\mathcal{U})$ of a family $\mathcal{U}$ of subspaces is a vector of length $n+1$ (indexed from 0 to n) with $f(\mathcal{U})_i = |\{U \in \mathcal{U} : \dim U = i\}|$, $i = 0, 1, ..., n$. In this paper we determine the profile polytope of intersecting families in the poset $L_n(q)$. A family $\mathcal{U}$ of subspaces is called intersecting if for any $U, U' \in \mathcal{U}$ we have $\dim(U \cap U') \geq 1$ (and t-intersecting if for any $U, U' \in \mathcal{U}$ we have $\dim(U \cap U') \geq t$). Two subspaces U, U' are said to be disjoint if $\dim(U \cap U') = 0$ i.e. $U \cap U' = \{\underline{0}\}$.

We will use the symbol $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)...(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)...(q-1)}$ for the Gaussian (q-nomial) coefficient denoting the number of k-dimensional subspaces of an n-dimensional space over $GF(q)$ (and q will be omitted, when it is clear from the context). The set of all k-dimensional subspaces of a vector space V will be denoted by $\begin{bmatrix} V \\ k \end{bmatrix}$.

In this section we determine the extreme points of the profile polytope of the set of intersecting families of subspaces. By Proposition 6 it is enough to determine the essential extreme points. This was implicitly done in [6] by Bey, but he only stated that his results concerning the Boolean lattice stay valid in the context of $L_n(q)$. What is more important, our approach is different from his: our main tool in determining some inequalities concerning the profile vectors of intersecting families of subspaces is Theorem 40. This is a generalization of a theorem of Hsieh ([25]) which might be of independent interest.

To simplify our counting arguments we introduce the following

Notation. If $k + d \leq n$, then $\begin{bmatrix} n \\ k \end{bmatrix}_q^{*(d)}$ denotes the number of k-dimensional subspaces of an n-dimensional vector space V over $GF(q)$ that are disjoint from a fixed d-dimensional subspace W of V.

Here are some basic facts about these numbers:

Facts.

$$\textbf{I.} \qquad \begin{bmatrix} n \\ k \end{bmatrix}^{*(d)} = \begin{bmatrix} n-d \\ k \end{bmatrix} q^{dk},$$

$$\textbf{II.} \qquad \frac{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}}{\begin{bmatrix} n \\ k \end{bmatrix}^{*(d)}} \leq \frac{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(n-k)}}{\begin{bmatrix} n \\ k \end{bmatrix}^{*(n-k)}} = \frac{1}{q^{n-k}} \leq \frac{1}{q^{k+1}} \qquad \text{(if } 2k+1 \leq n\text{)},$$

and so inductively for any $1 \leq p \leq k-1$

$$\textbf{III.} \qquad \frac{\begin{bmatrix} n-p \\ k-p \end{bmatrix}^{*(d)}}{\begin{bmatrix} n \\ k \end{bmatrix}^{*(d)}} \leq \frac{1}{q^{p(k+1)}} \qquad \text{(if } 2k+1 \leq n\text{)}.$$

The following theorem on intersecting families was first proved by Hsieh [25] (only for $n \geq 2k+1$) in 1977, then by Greene and Kleitman [24] (for the cases $k|n$ so especially if $n = 2k$) in 1978.

Theorem 39 *If $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ is an intersecting family of subspaces and $n \geq 2k$, then*

$$|\mathcal{F}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

The above theorem yields to the following inequalities concerning the profile vector of any intersecting family:

- $0 \leq f_i \leq \left[{n-1 \atop i-1} \right], \quad 0 \leq i \leq n/2$
- $0 \leq f_i \leq \left[{n \atop i} \right], \quad n/2 < i \leq n$

To establish more inequalities we will need the following statement:

Theorem 40 *The following generalization of Hsieh's theorem holds:*
(a) *if $2k \leq n \leq 2k+2$ and $d = 0$ or $d = n-k$*
or
(b) *if $n \geq 2k+3$ and $k+d \leq n$*

then for any intersecting family $\mathcal{F}$ of k-dimensional subspaces of an n-dimensional vector space V with all members disjoint from a fixed d-dimensional subspace U of V

$$|\mathcal{F}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}.$$

Note that the $d = 0$ case is just Hsieh's theorem.

Proof. If $k|d|n$ or $k|n$ and $d = 0$ then the argument of Greene and Kleitman [24] works. One can partition $V \setminus U$ into isomorphic copies of $V_k \setminus \{\underline{0}\}$, where V_k is a k-dimensional vector space over $GF(q)$. Since $\mathcal{F}$ cay contain at most one of the k-dimensional spaces of each partitioning set, the statement of the theorem follows.

So now we can assume $2k+1 \leq n$. We follow the argument in [25]. First we verify the validity of the lemmas from [25] in our context. For $x \in V$ $(A \leqslant V)$ let $\mathcal{F}_x$ $(\mathcal{F}_A)$ denote the set of subspaces in $\mathcal{F}$ containing x (A).

Lemma 41 (the analogue of Lemma 4.2. in [25]) *Suppose $n \geq 2k+1$ and let $\mathcal{F}$ be an intersecting family of k-subspaces of an n-dimensional space V such that all k-subspaces belonging to $\mathcal{F}$ are disjoint from a fixed d-dimensional subspace W of V (where $d \leq n-k$). If for all $x \in V$ we have $|\mathcal{F}_x| \leq \left[{n-1-p \atop k-1-p} \right]^{*(d)}$, then*

$$|\mathcal{F}| < \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)} \text{ or } |\mathcal{F}_A| \leq \begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix}^{*(d)} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p-1}$$

for all 2-dimensional subspaces A, where $1 \leq p \leq k-1$.

Proof. The statement in Lemma 4.2. in [25] is the same without the "star notation". There the proof uses only the estimate of the last "fact", and since this estimate remains valid with the "star notation", the proof goes through. ■

A more detailed version of this proof can be found in [23]. We will need one more lemma from Hsieh's paper (actualized to our context):

Lemma 42 (the analogue of Lemma 4.3. in [25]) *Let $\mathcal{F}$ be a family of intersecting k-subspaces of an n-dimensional space V of which all subspaces are disjoint from a fixed d-dimensional subspace W of V. Furthermore if*
(a) *$q \geq 3$ and $n \geq 2k+1$ and for all x we have $|\mathcal{F}_x| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{k-1}$,*
or if
(b) *$q = 2$ and*
$n \geq 2k+1$
and for all x we have $|\mathcal{F}_x| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{\min\{k-1, n-k-d\}} \prod_{i=1}^{k-1-(n-k-d)} \left(\begin{bmatrix} k \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right)$ (if $k-1 \leq n-k-d$, then the product is empty and equals 1),
then

$$|\mathcal{F}| < \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}.$$

Proof. In all cases $|\mathcal{F}|$ is at most $\begin{bmatrix} k \\ 1 \end{bmatrix}$ times the bound on $|\mathcal{F}_x|$.
Now if $q \geq 3$, then

$$|\mathcal{F}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^k = \left(\frac{q^k - 1}{q-1} \right)^k \leq q^{k^2-1} \leq q^{(k-1)(n-k)} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(n-k)} \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}.$$

If $q = 2$, then for any $n \geq 2k+1$ and $d = n-k$ we have

$$|\mathcal{F}| \leq \prod_{i=1}^{k-1} \left(\begin{bmatrix} k \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) < \begin{bmatrix} k \\ 1 \end{bmatrix}^{k-1} \left(\begin{bmatrix} k \\ 1 \end{bmatrix} - \begin{bmatrix} k-1 \\ 1 \end{bmatrix} \right) < (q^k)^{k-1} q^{k-1} =$$

$$q^{k^2-1} \leq q^{(k-1)(n-k)} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(n-k)}.$$

Since $n \geq 2k+1$, we have $n - 2k + 1 \geq 2$ holds. This gives

$$|\mathcal{F}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^k = \left(\frac{q^k - 1}{q-1} \right)^k < q^{2(k-1)} \frac{(q^{2k-2} - 1)(q^{2k-3} - 1)...(q^k - 1)}{(q^{k-1} - 1)(q^{k-2} - 1)...(q-1)} \leq$$

$$\leq q^{(k-1)(n-2k+1)} \begin{bmatrix} 2k-2 \\ k-1 \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(n-2k+1)}.$$

This establishes the lemma for $0 \leq d \leq n-2k+1$. For the remaining cases $(n-2k+1 < d < n-k)$ put $a_d = \left[\begin{smallmatrix} k \\ 1 \end{smallmatrix}\right]^{n-k-d+1} \prod_{i=1}^{k-1-(n-k-d)} \left(\left[\begin{smallmatrix} k \\ 1 \end{smallmatrix}\right] - \left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right]\right)$, $b_d = \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]^{*(d)}$. We have to prove that $\frac{a_d}{b_d} \leq 1$ holds for all $n-2k+1 < d < n-k$. To see this observe that

$$\frac{\frac{a_d}{b_d}}{\frac{a_{d+1}}{b_{d+1}}} = \frac{\left[\begin{smallmatrix} k \\ 1 \end{smallmatrix}\right]}{\left[\begin{smallmatrix} k \\ 1 \end{smallmatrix}\right] - \left[\begin{smallmatrix} n-k-d \\ 1 \end{smallmatrix}\right]} \cdot \frac{\left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]^{*(d+1)}}{\left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]^{*(d)}} = \frac{\left[\begin{smallmatrix} k \\ 1 \end{smallmatrix}\right]}{\left[\begin{smallmatrix} k \\ 1 \end{smallmatrix}\right] - \left[\begin{smallmatrix} n-k-d \\ 1 \end{smallmatrix}\right]} \cdot \frac{\left[\begin{smallmatrix} n-2-d \\ k-1 \end{smallmatrix}\right] q^{(d+1)k}}{\left[\begin{smallmatrix} n-1-d \\ k-1 \end{smallmatrix}\right] q^{dk}} =$$

$$\frac{q^k - 1}{q^k - q^{n-k-d}} \cdot \frac{q^{n-k-d} - 1}{q^{n-d-1} - 1} q^k \geq \frac{q^{n-k-d} - 1}{q^{n-d-1} - 1} q^k \geq 1.$$

Thus the sequence $\frac{a_d}{b_d}$ is monotone decreasing, and since $\frac{a_{n-2k+1}}{b_{n-2k+1}} \leq 1$ holds, so does $\frac{a_d}{b_d} \leq 1$ for all $n-2k+1 < d < n-k$.

This finishes the proof of the lemma. ■

Before we get into the details of the proof of Theorem 40, we just collect its main ideas:

the heart of the proof is the concept of *covering number*. For a family of *subsets* $\mathcal{F} \subseteq 2^{[n]}$ this is the size of the smallest set $S \subseteq [n]$ that intersect all sets in $\mathcal{F}$ (S need not be in $\mathcal{F}$). For a family of subspaces $\mathcal{F} \subseteq \left[\begin{smallmatrix} V \\ k \end{smallmatrix}\right]$ its covering number is the smallest number τ such that there is a τ-dimensional subspace U of V that intersects all subspaces that belong to $\mathcal{F}$. Observe that the proof of Lemma 41 was done by an induction on the covering number. The proof of Theorem 40 is again based on an induction on the covering number of $\mathcal{F}$. (During the proof, almost all computations will use the "facts" about Gaussian coefficients, all inequalities without any further remarks follow from them.)

If $x \in \cap\mathcal{F}$ for some $\underline{0} \neq x \in V$ then $|\mathcal{F}| \leq \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]^{*(d)}$. Thus we can suppose that $\cap\mathcal{F} = \{\underline{0}\}$.

Let $x_1 \neq \underline{0}$ be such that $|\mathcal{F}_{x_1}| = \max_{x \in V} |\mathcal{F}_x|$.

By our assumption, there is some $A_1 \in \mathcal{F}$ not containing x_1. Thus $|\mathcal{F}_{x_1}| \leq \left[\begin{smallmatrix} k \\ 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} n-2 \\ k-2 \end{smallmatrix}\right]^{*(d)}$.

Suppose that there are two independent vectors $z_1, z_2 \in A_1$ such that $A \in \mathcal{F} \Rightarrow A \cap \langle x_1, z_i \rangle \neq \{\underline{0}\}$ for $i = 1, 2$. If $u_i \in \langle x_1, z_i \rangle \setminus \langle x_1 \rangle$, then the u_i's are independent.

Thus

$$|\mathcal{F}| \leq |\mathcal{F}_{x_1}| + \sum_{U_i \subset (\langle x_1, z_i \rangle \setminus \langle x_1 \rangle) \cup \{\underline{0}\},\ dim(U_i)=1} |\mathcal{F}_{U_1,U_2}|$$

$$\leq \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}^{*(d)} + \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} - 1 \right)^2 \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}^{*(d)} < \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{(*(d))}.$$

Thus we can suppose that there is at most one $z \in A_1$ such that $A \in \mathcal{F} \Rightarrow A \cap \langle x_1, z \rangle \neq \{\underline{0}\}$. Suppose that $z \in A_1$ is such. Take $x \in A_1 \setminus \langle z \rangle$, then there is some $A \in \mathcal{F}$ such that $A \cap \langle x_1, x \rangle = \{\underline{0}\}$ and hence $|\mathcal{F}_{x_1,x}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}^{*(d)}$. Thus

$$|\mathcal{F}_{x_1}| \leq |\mathcal{F}_{x_1,z}| + \sum_{X \subset (A_1 \setminus \langle z \rangle) \cup \{\underline{0}\},\ \ dim(X)=1} |\mathcal{F}_{x_1,X}| \leq \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}^{*(d)} + \begin{bmatrix} k \\ 1 \end{bmatrix}^2 \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}^{*(d)}.$$

But then

$$|\mathcal{F}| \leq \sum_{X \subset \langle x_1, z \rangle, dim(X)=1} |\mathcal{F}_X| \leq \begin{bmatrix} 2 \\ 1 \end{bmatrix} \left(\begin{bmatrix} n-2 \\ k-2 \end{bmatrix}^{*d} + \begin{bmatrix} k \\ 1 \end{bmatrix}^2 \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}^{*(d)} \right) \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}.$$

Thus we can suppose that for all $x \in A_1$ there is some $A \in \mathcal{F}$ such that $A \cap \langle x_1, x \rangle = \{\underline{0}\}$, and hence $|\mathcal{F}_{x_1,x}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}^{*(d)}$. Thus $|\mathcal{F}_{x_1}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^2 \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}^{*(d)}$.

In general, suppose that for $1 \leq p \leq k-3$ we have non-zero vectors $y_1, y_2, ..., y_p \in V$ and $A_1, A_2, ..., A_{p+1} \in \mathcal{F}$ such that $y_i \in A$ and $A_{i+1} \cap \langle x_1, y_1, .., y_p \rangle = \{\underline{0}\}$ for $1 \leq i \leq p$. (We have just proved that either for any $y_1 \in A_1$ there exists such an $A_2 \in \mathcal{F}$ or the statement of the theorem holds.) Thus

$$|\mathcal{F}_{x_1,y_1,...,y_p}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)},$$

and so inductively we obtain that

$$|\mathcal{F}_{x_1}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}.$$

By Lemma 41, we have

$$|\mathcal{F}_{x,y}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{p} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}$$

for all 2-dimensional $\langle x, y \rangle \subset V$.

Suppose that there are $p+2$ linearly independent vectors $z_1, z_2, ..., z_{p+2}$ in A_{p+2} such that $\langle x_1, y_1, ..., y_p, z_i \rangle \cap A \neq \{\underline{0}\}$ for all $A \in \mathcal{F}$ and $i = 1, 2, ..., p+2$. Let $u_i \in$

$\langle x_1, y_1, ..., y_p, z_i \rangle \setminus \langle x_1, y_1, ..., y_p \rangle$, $i = 1, 2, ..., p+2$, then $u_1, u_2, ..., u_{p+2}$ are independent. Thus

$$|\mathcal{F}| \leq \sum_{X \subset \langle x_1, y_1, ..., y_p \rangle, dim(X)=1} |\mathcal{F}_X| + \sum_{U_i \subset (\langle x_1, y_1, ..., y_p, z_i \rangle \setminus \langle x_1, y_1, ..., y_p \rangle) \cup \{\underline{0}\}, dim(U_i)=1} |\mathcal{F}_{U_1, U_2, ..., U_{p+2}}|$$

$$\leq \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)} + \left(\begin{bmatrix} p+2 \\ 1 \end{bmatrix} - \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \right)^{p+2} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}$$

$$\leq \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)} + q^{(p+1)(k-1)} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}$$

$$\leq \left(\begin{bmatrix} p+1 \\ 1 \end{bmatrix} + 1 \right) \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*d} < \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}.$$

Thus we can suppose that there are at most $p+1$ such z_i's. Hence

$$|\mathcal{F}_{x_1, y_1, ..., y_p}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^2 \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix}^{*(d)} + \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)},$$

and so

$$|\mathcal{F}_{x_1}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+2} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix}^{*(d)} + \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}.$$

Suppose that we do have independent vectors $z_1, z_2 \in A_{p+2}$ such that $A \in \mathcal{F} \Rightarrow A \cap \langle x_1, y_1, ..., y_p, z_i \rangle \neq \{\underline{0}\}$ for $i = 1, 2$. Then

$$|\mathcal{F}| \leq \sum_{X \subset \langle x_1, y_1, ..., y_p \rangle, dim(X)=1} |\mathcal{F}_X| + \sum_{U_i \subset (\langle x_1, y_1, ..., y_p, z_i \rangle \setminus \langle x_1, y_1, ..., y_p \rangle) \cup \{\underline{0}\}, dim(U_i)=1} |\mathcal{F}_{U_1, U_2}|$$

$$\leq \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \left(\begin{bmatrix} k \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix}^{*(d)} + \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)} \right) +$$

$$+ \left(\begin{bmatrix} p+2 \\ 1 \end{bmatrix} - \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \right)^2 \begin{bmatrix} k \\ 1 \end{bmatrix}^{p} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}$$

$$= \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+2} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix}^{*(d)} + \left(\begin{bmatrix} p+2 \\ 1 \end{bmatrix}^2 + q^{2(p+1)} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p} \right) \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}$$

$$\leq \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+2} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix}^{*(d)} + q^{p} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}$$

$$\leq \left(\frac{\begin{bmatrix} p+1 \\ 1 \end{bmatrix}}{q^{p+2}} + \frac{1}{q} \right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)} < \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}.$$

Thus we can suppose that there is at most one such z. Hence

$$|\mathcal{F}_{x_1}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+2} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix}^{*(d)} + \begin{bmatrix} k \\ 1 \end{bmatrix}^{p} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}.$$

Suppose that $z_1 \in A_{p+1}$ is such a z, then

$$|\mathcal{F}| \leq \sum_{X \subset \langle x_1, y_1, \ldots, y_p, z_1 \rangle, dim(x)=1} |\mathcal{F}_X| \leq \begin{bmatrix} p+2 \\ 1 \end{bmatrix} \left(\begin{bmatrix} k \\ 1 \end{bmatrix}^{p+2} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix}^{*(d)} + \begin{bmatrix} k \\ 1 \end{bmatrix}^{p} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)} \right)$$

$$< \begin{bmatrix} p+2 \\ 1 \end{bmatrix} (\begin{bmatrix} k \\ 1 \end{bmatrix}^{p+2} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix}^{*(d)} + \frac{1}{q} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}^{*(d)}$$

$$\leq \left(\frac{\begin{bmatrix} p+2 \\ 1 \end{bmatrix}}{q^{p+2}} + \frac{1}{q^{p+2}} \right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)} < \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}.$$

Thus we can suppose that for all $z \in A_{p+1}$, there is some $A \in \mathcal{F}$ such that $A \cap \langle x_1, y_1, \ldots, y_p, z \rangle = \{\underline{0}\}$. Take $y_{p+1} \in A_{p+1}$, and let furthermore A_{p+2} be such that $A \cap \langle x_1, y_1, \ldots, y_p, y_{p+1} \rangle = \{\underline{0}\}$.

We obtained, that either the statement of the theorem holds, or there are linearly independent vectors $x_1, y_1, \ldots, y_{k-1}$ and $A_i \in \mathcal{F}$ $i = 1, \ldots k-1$ such that $y_i \in A_i$ and $\langle x_1, y_1, \ldots y_{i-1} \rangle \cap A_i = \{\underline{0}\}$. Furthermore we can suppose that y_i maximizes $|\mathcal{F}_{x_1, y_1, \ldots, y_{i-1}, z}|$ for $z \in A_i$.

If $q \geq 3$, this means that either $|\mathcal{F}| \leq \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]^{*(d)}$ or $|\mathcal{F}_x| \leq |\mathcal{F}_{x_1}| \leq \left[\begin{smallmatrix} k \\ 1 \end{smallmatrix}\right]^{k-1}$ and then we are done by Lemma 42.

If $q = 2$, we have to sharpen our estimates on $|\mathcal{F}_{x_1}|$. We know that for j independent vectors $x_1, y_1, \ldots, y_{j-1}$ with $U \cap \langle x_1, y_1, \ldots, y_{j-1} \rangle = \{\underline{0}\}$ there exists a subspace $A_j \in \mathcal{F}$ such that $A_j \cap \langle x_1, y_1 \ldots, x_{j-1} \rangle = \{\underline{0}\}$. Then we would have $|\mathcal{F}_{x_1, y_1, \ldots, y_{j-1}}| \leq \left[\begin{smallmatrix} k \\ 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} n-j-1 \\ k-j-1 \end{smallmatrix}\right]^{*(d)}$. Note that $U \cap \langle x_1, y_1, \ldots, y_{j-1} \rangle = \{\underline{0}\}$ must hold, as otherwise any subspace containing $x_1, y_1, \ldots, y_{j-1}$ would intersect U nontrivially, therefore $\mathcal{F}_{x_1, y_1, \ldots, y_{j-1}}$ would be empty, and thus, by the maximality assumption on the choice of y_{i-1}, $\mathcal{F}$ would be empty. Suppose further that for some positive l we have $j+k+d = n+l$. Then $\dim(\langle x_1, y_1 \ldots, y_{j-1}, A_j \rangle \cap U) \geq l$ and so (denoting $\langle x_1, y_1 \ldots, y_{j-1}, A_j \rangle \cap U$ by U_j) $\dim(\langle x_1, y_1 \ldots, y_{j-1}, U_j \rangle \cap A_j) \geq l$ as well, therefore when choosing among the vectors of A_j a subspace of dimension at least l is forbidden. Therefore we have the following better estimate on the number of subspaces in $\mathcal{F}$ containing $x_1, y_1, ..., y_{j-1}$:

$$\left(\begin{bmatrix} k \\ 1 \end{bmatrix} - \begin{bmatrix} l \\ 1 \end{bmatrix}\right) \begin{bmatrix} n-j-1 \\ k-j-1 \end{bmatrix}^{*(d)}.$$

Hence we have that either the statement of the theorem holds or the degree of any vector x is bounded by the expression given in the conditions of Lemma 42. So Lemma 42 establishes our theorem in this case, too. ■

Corollary 43 *For the profile vector f of any family $\mathcal{F}$ of intersecting subspaces of an n-dimensional vector space V, and for any $k < n/2$ and $n/2 < d \leq n-k$, the following holds*

$$c_{k,d} f_k + f_d \leq \begin{bmatrix} n \\ d \end{bmatrix},$$

where $c_{k,d} = q^d \frac{\begin{bmatrix} n-k \\ d \end{bmatrix}}{\begin{bmatrix} n-d-1 \\ k-1 \end{bmatrix}}$, and equality holds in case of $f_k = 0, f_d = \begin{bmatrix} n \\ d \end{bmatrix}$ or $f_k = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, f_d = \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}$.

Proof. Let us doublecount the disjoint pairs formed by the elements of $\mathcal{F}_k = \{U \in \mathcal{F} : \dim U = k\}$ and $\mathcal{F}'_d = \begin{bmatrix} V \\ d \end{bmatrix} \setminus \mathcal{F}_d = \{U \leqslant V, U \notin \mathcal{F} : \dim U = d\}$. On the one hand, for each $U \in \mathcal{F}_k$ there are exactly $q^{dk}\begin{bmatrix} n-k \\ d \end{bmatrix}$ such pairs (this uses the first *fact* about q-nomial coefficients), while on the other hand by Theorem 40 we know, that for any $W \in \mathcal{F}'_d$ there are at most $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)} = q^{d(k-1)}\begin{bmatrix} n-d-1 \\ k-1 \end{bmatrix}$ such pairs. This proves the required inequality and it is easy to see that equality holds in the cases stated in the Corollary. ■

Having established these inequalities, we are able to prove our main theorem.

Theorem 44 *The essential extreme points of the profile polytope of the set of intersecting families of subspaces are the vectors v_i $(1 \leq i \leq n/2)$ for even n and there is an additional essential extreme point v^+ for odd n, where*

$$(v_i)_j = \begin{cases} 0 & \text{if } 0 \leq j < i \\ \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} & \text{if } i \leq j \leq n-i \\ \begin{bmatrix} n \\ j \end{bmatrix} & \text{if } j > n-i, \end{cases} \tag{5.1}$$

and

$$(v^+)_j = \begin{cases} 0 & \text{if } \ 0 \le j < n/2 \\ \begin{bmatrix} n \\ j \end{bmatrix} & \text{if } \ j > n/2. \end{cases} \tag{5.2}$$

Proof. First of all, for any $x \in V$, for the families $\mathcal{G}_i = \{U : x \in U, i \le \dim U \le n-i\} \cup \{U : \dim U > n-i\}$ $(1 \le i \le n/2)$ $f(\mathcal{G}_i) = v_i$ holds, and if n is odd then the profile of the family $\mathcal{G}^+ = \{U : \dim U > n/2\}$ is v^+, and clearly none of these vectors can be dominated by any convex combination of the others.

We want to dominate the profile vector f of any fixed intersecting family $\mathcal{F}$ with a convex combination of the vectors v_j (and possibly v^+ if n is odd). We define the coefficients of the v_js recursively. Let i denote the index of the smallest non-zero coordinate of f. For all $j < i$ let $\alpha_j = 0$. Now if for all $j' < j$ $\alpha_{j'}$ has already been defined, let

$$\alpha_j = \max\left\{ \frac{f_j}{\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}} - \sum_{j'=i}^{j-1} \alpha_{j'}, 0 \right\}.$$

Note, that for all j $(i \le j \le n/2)$ the jth coordinate of $\sum_{j'=i}^{j} \alpha_{j'} v_{j'}$ is at least f_j (and equality holds if when choosing α_j, the first expression is taken as maximum), so these vectors already dominates the "first part" of f.

When all α_js $(i \le j \le n/2)$ are defined, then let $\alpha^+ = 1 - \sum_{j'=i}^{n/2} \alpha_{j'}$ and let α^+ be the coefficient of v^+ if n is odd or add α^+ to the coefficient of $v_{n/2}$ if n is even. Note also that α^+ is non-negative since for all $i \le j \le k \le n/2$ $(v_j)_k = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ and by Hsieh's theorem $0 \le f_k \le \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$. Therefore this is really a convex combination of the v_js.

The easy observation that this convex combination dominates f in the coordinates larger then $n-i$ follows from the fact that all v_js (and v^+ as well) have $\begin{bmatrix} n \\ d \end{bmatrix}$ in the dth coordinate, therefore so does the convex combination which is clearly an upper bound for f_d.

All what remains is to prove the domination in the dth coordinates for all $n/2 < d \le n-i$, that is to prove the inequality

$$f_d \le \begin{bmatrix} n-1 \\ d-1 \end{bmatrix} \sum_{j=i}^{n-d} \alpha_j + \begin{bmatrix} n \\ d \end{bmatrix} (1 - \sum_{j=i}^{n-d} \alpha_j).$$

Let $k \le n-d$ be the largest index with $\alpha_k > 0$. Then we have

$$f_d \le \begin{bmatrix} n \\ d \end{bmatrix} - c_{k,d} f_k = \begin{bmatrix} n \\ d \end{bmatrix} - c_{k,d} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \sum_{j=i}^{k} \alpha_j = (1 - \sum_{j=i}^{k} \alpha_j) \begin{bmatrix} n \\ d \end{bmatrix} + \begin{bmatrix} n-1 \\ d-1 \end{bmatrix} \sum_{j=i}^{k} \alpha_j$$

$$= (1 - \sum_{j=i}^{n-d} \alpha_j) \begin{bmatrix} n \\ d \end{bmatrix} + \begin{bmatrix} n-1 \\ d-1 \end{bmatrix} \sum_{j=i}^{n-d} \alpha_j,$$

where the inequality is just Corollary 43, the first equality follows from the fact that $\alpha_k > 0$, the second equality uses again Corollary 43 (the statement about when equality holds) and the last equality uses the defining property of k (for all $k < j \le n-d$ $\alpha_j = 0$). This proves the theorem. ■

Note that, the (essential) extreme points are analogous to the Boolean case, one just has to change the binomial coefficients to the corresponding q-nomial coefficients.

Chapter 6

Finding the maximum and minimum elements with one lie

This chapter is based on our joint work with Dömötör Pálvölgyi, Balázs Patkós and Gábor Wiener ([21]).

The minimum number of comparisons needed in order to find both the maximal and minimal elements is $\lceil \frac{3n}{2} \rceil - 2$ if all answers have to be correct (see [33]). It can be easily seen using the following observation: if at the beginning of the algorithm we give each x_i a red and a blue pebble and after each comparison we remove the red pebble of the smaller element and the blue pebble of the larger element (if they still possess it), then the algorithm terminates if and only if we have only one element having a red pebble (the largest element) and another element having a blue pebble (the smallest element).

One could think that if k erroneous answers are allowed, then all one has to do is to use $k+1$ red and blue pebbles instead of one, as if an element has been said to be the larger one in $k+1$ comparisons, then in at least one of these it was indeed the larger and hence cannot be the smallest element. Unfortunately an if-and-only-if-type statement does not hold now, but before explaining this let us introduce some notations and the "soccer terminology".

The element x_i will be called the ith *team*, a comparison will be called a *match* which is a *win* for the team of the larger element and a *loss* for the team of the smaller element. We will also say that x_i beats x_j if the match between x_i and x_j

ended with a win for x_i. For a team x, let $w(x)$ denote the number of wins of x, and let $l(x)$ be the number of losses of x. In the case of k erroneous answers, we put $wl_k(x) = (\max\{k+1-w(x), 0\}, \max\{k+1-l(x), 0\})$, the number of wins and losses that are still needed in order to prove that x is neither the maximal nor the minimal element. We also use the notation $a^+ = \max\{a, 0\}$, so for example we can write $wl_k(x) = ((k+1-w(x))^+, (k+1-l(x))^+)$.

Let us define the *championship graph* G as follows: the vertex set of this directed multigraph is the set of teams, and for each match a directed edge is given to the graph oriented from the loser toward the winner.

If the championship graph contains a directed cycle, then we know that for one of the matches corresponding to the edges of the cycle we were given an erroneous result. Therefore if we forget about the results corresponding to the edges of the cycle, we know that among the other results (including the forthcoming ones) there can be at most $k-1$ lies. This is the reason why the above-mentioned if-and-only-if-type statement is not true in this case. However, the obvious direction still holds as stated in the following claim.

Proposition 45 *If at most k erroneous answers are allowed, then a team x with $wl_k(x) = (0, 0)$ cannot be the maximum or the minimum element.*

Corollary 46 *Suppose that at most k erroneous answers are allowed and we have exactly two elements x with $wl_k(x) \neq (0, 0)$. If for both of these elements either the number of losses or the number of wins is k, then they are the extremal elements.*

Corollary 46 will serve to prove upper bounds on the number of comparisons needed to find the extremal elements in different models. To provide lower bounds we will use the notion of an *Adversary*. A strategy of an Adversary is a function that tells us what the Adversary answers for a query in the view of previous queries and answers. To obtain lower bounds we will have to prove that there exists an Adversary's strategy that answers any sequence of queries in such a way that until at least D comparisons asked, no strategy of queries determines both the maximum and the minimum elements. How can one guarantee that a sequence of queries and answers does not determine the extremal elements? Observe that a championship graph may consist of true answers if and only if it is acyclic. Furthermore, it is obvious that if in a directed acyclic graph G one changes the orientation of all incoming (outgoing) edges that are adjacent to a fixed vertex v, then the resulting graph G' is also acyclic. These two easy observations give us the following Corollary.

Corollary 47 *If at most k erroneous answers are allowed, and if there exists a strategy of an Adversary that can assure that after D queries the championship graph is acyclic and there exists at least two vertices either both with in-degree at most k or both with out-degree at most k, then the number of comparisons needed to find the maximum and the minimum elements is at least $D+1$.*

The rest of the chapter is organized as follows: in Section 6.1 we present a simple (and not optimal) algorithm and bound the number of comparisons it uses for arbitrary k. This algorithm was described already by Aigner in [3], but Aigner's proof for the number of comparisons used in the algorithm is somewhat different from ours and the method of our proof is used later to give an almost matching lower bound in the case $k = 1$. In Section 6.2, we address the original problem with at most one lie allowed and prove the following main result.

Theorem 48 *For the minimum number $M(n)$ of comparisons needed to find the extremal elements among n elements if there might be one erroneous answer, we have*

$$\lceil \frac{87n}{32} \rceil - 3 \leq M(n) \leq \lceil \frac{87n}{32} \rceil - 2.$$

Aigner in [3] stated the upper bound and conjectured it to be optimal, thus Theorem 48 verifies his conjecture. In Section 6.3 we gather some concluding remarks.

6.1 Algorithm for arbitrary k

In this section we give an algorithm that does not use the possible additional information that might be gained from the existence of directed cycles in the championship graph. First let us introduce a slightly different version of the problem, when the algorithm "cannot use" this additional information.

We are given n teams $x_1, ..., x_n$ and every team x_i possesses an ordered pair $wl_k(x_i) = (a_i, b_i)$. At the beginning of the procedure $a_i = b_i = k+1$ for all $1 \le i \le n$. A query in this version is a pair of teams $\{x_i, x_j\}$ and there are two possible "answers": either $wl_k(x_i) = ((a_i-1)^+, b_i), wl_k(x_j) = (a_j, (b_j-1)^+)$ or $wl_k(x_i) = (a_i, (b_i-1)^+), wl_k(x_j) = ((a_j-1)^+, b_j)$ but there must always be a team with a positive a_i and another one with a positive b_i. The process ends when all but two (a_i, b_i) pairs are $(0,0)$ and from the remaining two, at least one has a zero a_i or b_i. Denote the minimum number of queries needed to obtain this situation by $N(k,n)$. In the remainder of this section we will prove the following theorem.

Theorem 49

$$N(k,n) = (k+1)(1+\binom{2(k+1)}{k+1}2^{-2(k+1)})n + \Theta_k(1) = (k+\Theta(\sqrt{k}))n + \Theta_k(1).$$

It is clear that any upper bound on $N(k,n)$ is also an upper bound for the number of comparisons needed in the original problem, since every algorithm that solves this problem, also solves the original one because of Corollary 46.

Proof. We define a symmetric potential function $p : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Let $p(a,0) = p(0,a) = a$ for any $a \in \mathbb{N}$ and let us define the other values recursively by the equation

$$2p(a,b) = p(a-1,b) + p(a,b-1) + 1. \tag{6.1}$$

Now we determine the value $p(k,k)$. Putting $g(a,b) = 2^{a+b}p(a,b) - (a+b)2^{a+b-1}$, equation (1) transforms to

$$g(a,b) = g(a-1,b) + g(a,b-1) \tag{6.2}$$

with $g(a,0) = a2^{a-1}$. Here we see the same recursion as for the binomial coefficients, but unfortunately the initial values differ. For $a, b > 0$ we have

$$g(a,b) = \sum_{i=1}^{a} g(i,0)\binom{a-i+b-1}{b-1} + \sum_{j=1}^{b} g(0,j)\binom{a-1+b-j}{a-1}.$$

From this we can determine the value of $g(k,k)$.

$$g(k,k) = 2\sum_{i=1}^{k} g(i,0)\binom{2k-1-i}{k-1} = \sum_{i=1}^{k} i2^i\binom{2k-1-i}{k-1}.$$

This can be transformed into a nice, explicit form using properties of binomial coefficients.

Lemma 50 $\sum_{i=1}^{k} i2^i \binom{2k-1-i}{k-1} = \binom{2k}{k} k$.

Proof.

$$2\sum_{i=1}^{k} i2^{i-1}\binom{2k-1-i}{k-1} = 2\sum_{i=1}^{k}\binom{2k-1-i}{k-1}\cdot\sum_{j=0}^{i} i\binom{i-1}{j} =$$

$$2\sum_{i=1}^{k}\binom{2k-1-i}{k-1}\cdot\sum_{j=0}^{i} j\binom{i}{j} = 2\sum_{j=1}^{k} j\sum_{i=j}^{k}\binom{i}{j}\binom{2k-1-i}{k-1}.$$

For the inner part we have

$$\sum_{i=j}^{k}\binom{i}{j}\binom{2k-1-i}{k-1} = \binom{2k}{k+j},$$

because both sides count the number of 0–1 sequences of length $2k$ with $k+j$ 1-coordinates. (Each part of the sum on the left hand side counts the sequences in which the $(j+1)$st 1 is in the $(i+1)$st position.) Using this we obtain

$$2\sum_{j=1}^{k} j\sum_{i=j}^{k}\binom{i}{j}\binom{2k-1-i}{k-1} = 2\sum_{j=1}^{k} j\binom{2k}{k+j} =$$

$$2\left(\sum_{j=1}^{k}(k+j)\binom{2k}{k+j} - \sum_{j=1}^{k} k\binom{2k}{k+j}\right) =$$

$$2\left(2k\sum_{j=1}^{k}\binom{2k-1}{k+j-1} - k\sum_{j=1}^{k}\binom{2k}{k+j}\right) =$$

$$2\left(k2^{2k-1} - (k2^{2k-1} - \frac{1}{2}k\binom{2k}{k}))\right) = k\binom{2k}{k}. \ \blacksquare$$

This implies $p(k,k) = k + g(k,k)/2^{2k} = k(1 + \binom{2k}{k}/2^{2k})$.

Put $p(x) = p(wl_k(x))$ and observe the following:

1. If a query involves x and y with $wl_k(x) = wl_k(y) \neq (0,0)$, then because of (1) the sum $\sum_{i=1}^{n} p(x_i)$ decreases by exactly 1. Until at most $(k+1)^2$ teams remain with $wl_k(x) \neq (0,0)$, we can always find such a query by the pigeonhole principle, therefore

we obtain our desired situation using at most $p(k+1,k+1)n+c_k$ queries, which gives the upper bound of the theorem.

2. If a query involves teams x and y with $wl_k(x)=(a,b)$, $wl_k(y)=(c,d)$, then the possible “outcomes” are $wl_k(x)=((a-1)^+,b)$, $wl_k(y)=(c,(d-1)^+)$ and $wl_k(x)=(a,(b-1)^+)$, $wl_k(y)=((c-1)^+,d)$. Again by (1), it is clear that the decrease of $\sum_{i=1}^n p(x_i)$ is 2 if we add up the decrease of both possible cases, so with one of the possible outcomes this sum will decrease by at most 1. If the Adversary's strategy is to answer all queries in such a way that the sum decreases by at most 1, then it is obvious that one needs at least $p(k+1,k+1)n-p(k+1,0)-p(k+1,k+1)\geq p(k+1,k+1)n-3k-3$ queries, which gives the lower bound of the theorem. ■

6.2 Selection with one lie

In this section we prove Theorem 48. In the first subsection we describe an optimal algorithm (within an additive constant) to find the maximum and minimum elements despite at most one erroneous answer. In the second subsection we modify the potential function used in the proof of Theorem 49 to prove the lower bound in Theorem 48. We use the notation $wl(x)=wl_1(x)=((2-w(x))^+,(2-l(x))^+)$.

Proposition 51 *A question such that both possible answers result in removing a pebble can always be asked, unless we are done.*

Corollary 52 *If there are r pebbles, and there exists a team without losses and another team without wins, then $r-3$ questions suffice to finish the algorithm.*

Proof. We always ask questions that remove at least 1 pebble. At the end of the algorithm there may be a directed cycle but there is at most one erroneous result hence there is at least one edge e such that all the directed cycles disappear by removing this edge. Thus e can be the only loss of the maximum and the only win of the minimum. Hence all the other answers could not remove the red pebbles of the minimum and the blue pebbles of the maximum, hence they removed at most $r-4$ pebbles. So there were at most $r-4$ questions without e.■

6.2.1 Upper bound

In this subsection we describe an algorithm that finds the smallest and the largest elements of a set of size n using not more than $(11/4 - 1/32)n + 3$ comparisons if at most one of the comparisons may turn out to be erroneous. Note that the algorithm of the previous section only gives an algorithm that uses $11n/4 + O(1)$ questions. Let $n = 32m + q$, where $0 \leq q \leq 31$. At first we suppose $q = 0$.

We describe our algorithm in rounds. A round is a set of matches that can be played at the same time. In the first round we consider an arbitrary maximum matching of the teams, therefore with $n/2$ matches played we will have a set X of $n/2$ teams with $wl(x) = (1, 2)$ for all $x \in X$ and a set Y of $n/2$ teams with $wl(y) = (2, 1)$ for all $y \in Y$. In the second round we consider a maximum matching of the teams of X. With this additional $n/4$ matches X will be divided into X_1 and X_2 such that $|X_1|, |X_2| = n/4$ and $wl(x_1) = (0, 2)$ for all $x_1 \in X_1$ and $wl(x_2) = (1, 1)$ for all $x_2 \in X_2$.

In the third round of our algorithm we divide X_2 into two using a matching of $n/8$ additional matches. We obtain $X = X_1 \cup X_2^1 \cup X_2^2$ with $|X_2^1|, |X_2^2| = n/8$ and $wl(x) = (0, 1)$ for all $x \in X_2^1$ and $wl(x) = (1, 0)$ for all $x \in X_2^2$.

Let Y' be the set of teams in Y that were matched in the first round with teams in X_2^2. The fourth round of our algorithm consists of a matching of Y such that any team of Y' plays another team from Y' (i.e. we use a matching of Y that is an expansion of a matching of Y'). After the $n/4$ matches of the fourth round we will have $Y = Y_1 \cup Y_2$ with $|Y_1|, |Y_2| = n/4$, $|Y_2 \cap Y'| = n/16$ and $wl(y) = (2, 0)$ for all $y \in Y_1$ and $wl(y) = (1, 1)$ for all $y \in Y_2$.

In the fifth round of our algorithm we use a matching of Y_2 that is an expansion of a matching of $Y_2 \cap Y'$. After these $n/8$ matches we will have $Y = Y_1 \cup Y_2^1 \cup Y_2^2$ with $|Y_2^1| = |Y_2^2| = n/8$, $|Y_2^2 \cap Y'| = n/32$ and $wl(y) = (1, 2)$ for all $y \in Y_2^1$ and $wl(y) = (2, 1)$ for all $y \in Y_2^2$.

The sixth round is where our algorithm gains the extra $n/32$ matches. In this round the matches that were played in the first round between teams of $Y_2^2 \cap Y'$ and their opponents all, are replayed. Remember those matches were won by the teams in X_2^2. If for all matches the same results are obtained as in the first round then for any team x involved in this round we have $wl(x) = (0, 0)$. In this case after the $n/2+n/4+n/8+n/4+n/8+n/32 = 41n/32$ matches of the first six rounds we will have

$n/16$ teams with $wl(x) = (0,0)$, the number of teams with $wl(x) = (0,2)$ or $(2,0)$ is $n/4$ each, while the number of teams with $wl(x) = (0,1)$ or $(1,0)$ is $7n/32$ each. The total number of the remaining pebbles is (at most) $2\cdot(n/4+n/4)\cdot(7n/32+7n/32) = 46n/32$, hence by Corollary 52 we can finish the algorithm using $46n/32-3$ questions. Thus the total number of matches played during our algorithm is at most $41n/32+46n/32-3 = 87n/32-3$.

Now we consider what happens if any match in round six ends with a different result than it ended in round one (since only one lie is allowed, there can be at most one such match). In this case we do not know the "real result" of this match, but we know that the results of all the other matches (including the forthcoming ones) are correct, so deleting the two contradicting scores leaves us in finding the smallest and the largest element *without* lies. Therefore for every team x, we can replace $wl(x) = (a,b)$ by $wl_0(x) = ((a-1)^+, (b-1)^+)$. In this way, after the $41n/32$ matches of the first six round, all teams x have $wl_0(x) = (1,0), (0,1)$ or $(0,0)$, therefore we can finish our algorithm with at most n queries which gives a total of $73n/32$ queries.

If $q \neq 0$, we use this algorithm on $32m$ elements. It uses at most $87m-3$ queries (except the case $m = 0$, when it uses $0 = 86m$ queries). After that only the maximum and the minimum can have any pebbles, plus the q additional elements. Hence there are at most $2+2+4q$ pebbles, hence by Corollary 52 the algorithm can be finished using $4q+1 \leq 125$ questions. It proves the upper bound of Theorem 48 with $c = 122$.

Actually one can deal with the q elements in a smarter way and easily see that the upper bound in Theorem 48 is also true with $c = -2$. We omit the technical details here. ■

6.2.2 Lower bound

In this subsection we describe a strategy for the Adversary which shows that at least $87n/32-3$ queries are necessary to find both the maximum and the minimum elements. Because of the observation made in the introduction, this strategy should avoid making directed cycles in the championship graph until the very end of the algorithm. We will use a potential function p just as in Section 6.1, but as the answer for this problem is different from that of the problem in Section 6.1 we have to modify this function a bit using a "correction function" c. For convenience's sake we first enumerate the values of $p(x) = p(wl(x))$ that we need: $p(0,0) = 0$, $p(1,0) = p(0,1) = 1$, $p(2,0) = p(0,2) = 2$, $p(1,1) = 1.5$, $p(2,1) = p(1,2) = 2.25$, $p(2,2) = 2.75$. Note that if any x and y play each other, then there is a possible outcome such that $p(x)+p(y)$ decreases by at most one.

The function c is defined for each ordered pair of teams, including the case when the two teams are the same.

Let us define $c(x,x) = 1/32$ if $wl(x) = (2,2)$, i.e. if the team x has not played any matches yet, and $c(x,x) = 0$ if $wl(x) \neq (2,2)$.

Let x and y be two distinct teams. If x and y has played their very first game against each other, x has beaten y, and since that x has not won and y has not lost any matches, then x and y are said to be pairs of each other. If this is the case, then let $c(x,y) = (1/2)^{(2-l(x))^+ + (2-w(y))^+}$, otherwise let $c(x,y) = 0$.

With this modification the Adversary will have a strategy avoiding directed cycles such that $\sum p(x) - \sum c(x,y)$ decreases by at most 1 after each comparison. At the beginning of the algorithm $\sum p(x) - \sum c(x,y) = n(p(2,2) - 1/32) = 87n/32$ and at the end of the algorithm this sum is at most $p(2,0) + p(0,2) = 4$, thus Corollary 1.3 gives $87n/32 - 4$ as a lower bound (at the end of the subsection we strengthen this bound by 1 to obtain the statement of the theorem).

Now we define some special subsets of teams that will change during the game. The *Champions League* and the *Second Division* are both empty at the beginning and if a team becomes an element of one of them, it stays there forever. After each comparison, a team becomes an element of the Champions League if it is not yet in the Second Division, it was only beaten by teams who are now in the Champions League and it has two wins. Similarly, after each comparison, a team becomes an element of the Second Division if it is not yet in the Champions League, it only won against teams who are now in the Second Division and it has two losses. Note that not only the winner (loser) of a comparison may move into the Champions League (Second Division), e.g. if $wl(x) = (1,0)$ and the only team beaten by x moves into the Second Division, then x moves there as well. If a team is not an element of the Champions League or the Second Division, we say that it is *active*. We say that an active team is *in reach* of the Champions League (or of the Second Division) if it only needs one more win (loss) to become a member. We would like to find an Adversary's strategy such that during the whole process every team that has already played a game, is either a member of the Champions League or of the Second Division or is in reach of (at least) one of them (condition 1). Furthermore, every previous opponent of each active team will be inactive except maybe its pair (if it has any) (condition 2).

Now we describe the strategy of the Adversary, that is, we exhibit a function that decides who is winning which game such that $S = \sum p(x) - \sum c(x, y)$ decreases by at most 1 after each comparison and the above mentioned conditions hold.

If a team gets into the Champions League, then from that on it will win every match against teams that were not in the Champions League at the moment of its qualification (i.e. the moment when it became a member of the Champions League). Similarly, if a team gets into the Second Division, then it will lose every further match against teams that were not in the Second Division at the moment when it got there. Obviously, this kind of matches cannot give directed cycles.

If two active, pairless teams play, then there always exists an answer that decreases S by at most 1. This answer cannot give a directed cycle since all their previous opponents were already inactive. Also note that, unless this was the first game for both teams, one of the teams becomes inactive.

The only case that remains is when an active team x who has an active pair y is playing another active team z. Without loss of generality, suppose that x has beaten y in their first game. By condition 1, this implies that x is in reach of the Champions' League and y is in reach of the Second Division. The possible values of $wl(x)$ are $(1, 2), (1, 1)$ and $(1, 0)$, while the possible values of $wl(z)$ are $(2, 1), (1, 1)$ and $(0, 1)$.

Case 0: $z = y$. To avoid a cycle of length two, x has to win the game. S decreases by $p(x) + p(y) - c(x, y)$ (since $c(x, y)$ vanishes after the game) and it is easy to check that this is at most 1.

Case 1: z has no pair. This means that z cannot have two wins or losses (otherwise it would not be active).

CASE 1.1: $l(z) \geq w(z)$.

CASE 1.1.1: $wl(x) \neq (1,0)$. If x wins, then $p(z)$ decreases by at most 0.5, $p(x)$ also decreases by at most 0.5, $c(x,y)$ vanishes.

CASE 1.1.2: $wl(x) = (1,0)$.

CASE 1.1.2.1: $wl(z) = (1,1)$. If z wins, it moves into the Champions' League, x and y remain unaffected.

CASE 1.1.2.2: $wl(z) = (2,1)$. If x wins, $p(z)$ decreases by 0.25, $p(x)$ decreases by 1, but $c(x,y) \geq 1/4$ vanishes, since x moves into the Champions League.

CASE 1.1.2.3: $wl(z) = (2,2)$. Now we need z to win but this would not make it move into the Champions League ruining condition 2. We solve this problem by giving some more information: we answer the same question again without being asked, this way $wl(z)$ becomes $(0,2)$ and z moves into the Champions League. Of course we are not allowed to count the question twice, but we do not need to if we can show that S decreases by at most 1 after the two answers. Indeed, $p(z)$ only decreases by 0.75, while x and y are unaffected.

CASE 1.2: $l(z) < w(z)$. This means that $wl(z) = (1,2)$.

CASE 1.2.1: $wl(x) = (1,0)$. If z wins, it moves into the Champions League, x and y remain unaffected.

CASE 1.2.2: $wl(x) = (1,2)$. If x wins, it moves into the Champions League, $p(z)$ decreases by 0.75, $p(x)$ decreases by 0.25, $c(x,y)$ vanishes.

CASE 1.2.3: $wl(x) = (1,1)$.

CASE 1.2.3.1: $c(x,y) \geq 1/4$. If x wins, it moves into the Champions League, $p(z)$ decreases by 0.75, $p(x)$ decreases by 0.5, $c(x,y)$ vanishes.

CASE 1.2.3.2: $c(x,y) = 1/8$. If z wins, it moves into the Champions League, $p(z)$ decreases by 0.25, $p(x)$ decreases by 0.5, $c(x,y)$ increases by 1/8.

CASE 2: z has a pair q who was beaten by z. Now either x or z moves into the Champions' League and either $c(x,y)$ or $c(z,q)$ vanishes. Note that in this case the roles of x and z are symmetric, this eliminates some cases.

CASE 2.1: $wl(x) = wl(z)$. If $c(x, y) \leq c(z, q)$, then z wins, otherwise x wins, so $p(x) + p(z)$ decreases by 1, $c(x, y) + c(z, q)$ does not increase.

CASE 2.2: $wl(x) = (1, 0)$. If z wins, $p(z) - c(z, q)$ decreases by at most 1, x and y are unaffected.

CASE 2.2': $wl(z) = (1, 0)$ is analogous to 2.2.

CASE 2.3: $wl(x) = (1, 1)$, $wl(z) = (1, 2)$.

CASE 2.3.1: $c(x, y) \leq c(z, q) + 1/4$. If z wins, $p(x) + p(z)$ decreases by 0.75, $c(x, y)$ increases by at most $c(z, q) + 1/4$, while $c(z, q)$ vanishes.

CASE 2.3.2: $c(x, y) = 1/2$, $c(z, q) < 1/4$. If x wins, $p(x) + p(z)$ decreases by 1.25, $c(z, q)$ increases by less than 1/4, while $c(z, q)$ vanishes.

CASE 2.3': $wl(x) = (1, 2)$, $wl(z) = (1, 1)$ is analogous to 2.3.

CASE 3: z has a pair q who has beaten z.

CASE 3.1: $wl(z) = (2, 1)$. If x wins, $p(z)$ decreases by 0.25 and $p(x) - c(x, y)$ decreases by at most 0.75.

CASE 3.2: $wl(z) = (1, 1)$.

CASE 3.2.1: $wl(x) \neq (1, 0)$. If x wins, $p(z)$ decreases by 0.5, $p(x)$ also decreases by at most 0.5, $c(x, y)$ and $c(q, z)$ vanish.

CASE 3.2.2: $wl(x) = (1, 0)$.

CASE 3.2.2.1: If $c(x, y) + c(q, z) \geq 1/2$, then let x win, so $p(z)$ decreases by 0.5, $p(x)$ decreases by 1, but $c(x, y)$ and $c(q, z)$ vanish.

CASE 3.2.2.2: If $c(x, y) + c(q, z) < 1/2$, then $c(x, y) = 1/4$ and $c(q, z) = 1/8$, thus we have $wl(q) = (1, 2)$ and $wl(y) = (2, 1)$. If z wins, then $p(z)$ decreases by 0.5 and

$c(q,z)$ increases by 1/8. Condition 2 is ruined, so we give some more information just like in case 1.1.2.3. We give the additional information that q has beaten y, making all the involved teams inactive. Now $p(q)$ and $p(y)$ decrease by 0.25 each, while $c(x,y)$ and $c(q,z)$ vanish.

CASE 3.3: $wl(z) = (0,1)$.

CASE 3.3.1: If $wl(x) \neq (1,0)$, then this is the same situation as 3.1 or 3.2.2, just swap the roles of x and z and the wins and losses.

CASE 3.3.2: $wl(x) = (1,0)$. If z wins, then S remains unchanged, but condition 2 is ruined. We again use the trick of giving unwanted information, we say that q has beaten z (for a second time). This way they both go into the Champions League and S decreases by at most 1.

We checked all the cases, which proves the bound $M(n) \geq 87n/32 - 4$.

Now we show how to strengthen this bound by 1 to match the lower bound of Theorem 48.

According to the Adversary's strategy we have described, in the very last match either a team with only one win wins (so it cannot be the minimum) or a team with only one loss loses (so it cannot be the maximum). Now we change the answer of the Adversary to this last question. We claim that in this way either the minimum or the maximum element remains unknown, hence another question is needed, which proves the lower bound of Theorem 48. We may suppose that a team x with only one win is beaten by a team y. Now we have two different possibilities to make the championship graph acyclic by changing the orientation of at most one edge: either we change the edge corresponding to this last match or we change the edge corresponding to the match won by x earlier. It is easy to see that the minimum elements are different for the two cases, thus we need (at least) one more question to find the minimum element. ■

6.3 Further results and remarks

In this final section we gather further results related to Theorem 48.

In what follows, we enumerate some models where either restrictions are posed for the possible comparisons or for the relation of the possible erroneous answers. One way that a restriction can be posed is if one can ask a pair $\{x_i, x_j\}$ to be compared at most once. We call this restricted model the *Gentlemen's model*.

With this restriction one cannot find the maximum provided one lie is allowed even if every possible pair is compared. To see this, just observe that if the one and only erroneous answer is when the maximum is compared to the third largest element, then clearly one cannot tell the difference between the three largest elements.

However, one can find algorithms that provide solutions for the following problems:

(i) Find 3 elements such that one of them is the largest.

(ii) Find an element which is one of the three largest.

The next theorem gives the exact solution for the first problem.

Theorem 53 *In the Gentlemen's model the minimum number of comparisons needed to find 3 elements such that one of them is the largest is* $2n-5$*, if* $n>3$.

Proof. First we describe the optimal algorithm.

STEP 1: The teams x_1 and x_2 play a match. Let x_1 be the loser.

STEP 2: Two teams without a match play. Denote the loser by x_3.

STEP 3: x_1 and x_3 play.

STEP 4: Delete the loser of the previous match and the two matches it has lost. Denote the winner of the previous match by x_1, since it is the loser of the only remaining edge. Go to STEP 2.

We continue this procedure until there are only 3 undeleted elements.

In every execution of STEP 2 an element is deleted that cannot be the largest because it has two losses. Therefore one of the remaining 3 elements is the largest. There are $n-3$ elements deleted, hence STEP 4, STEP 2 and STEP 3 are executed

$n-3$ times and STEP 1 only once. Comparisons occur once in STEP 1, STEP 2 and STEP 3, so there are at most $2n-5$ of them.

For the lower bound we describe an Adversary's strategy. The order of the elements will be determined after the first question and the Adversary will never lie. The winner of the first match will be the largest element, the loser the second largest and fix an arbitrary order for the rest.

Clearly there will be no directed cycles in the championship graph. Hence an element x can be the largest one if and only if it has lost at most one match. When someone names three elements such that the largest element is among them, all the other $n-3$ elements must have lost at least two matches. We also know that the second largest lost exactly one match, so there have been at least $2n-5$ matches. This finishes the proof. ■

We mention an upper bound for the second problem without proof:

Theorem 54 *In the Gentlemen's model the minimum number of comparisons needed to find an element which is one of the three largest is at most* $2n - \log n + O(1)$.

Problems that we dealt with in Section 6.1 and 6.2 were about to find the maximum and the minimum element. In the Gentlemen's model, we cannot ask for an algorithm that would provide us these elements, but we could ask for an algorithm that gives 6 elements that contain the maximum and the minimum (in fact, 4 elements would suffice). Note that the algorithm presented in Theorem 49 can be arranged in such a way that no comparisons are asked twice, therefore $11/4n$ is a trivial upper bound and the lower bound $(11/4-1/32)n$ of Theorem 48 obviously remains valid in the more restrictive Gentlemen's model. Again we state a better upper bound without proof that we conjecture to be (asymptotically) optimal.

Theorem 55 *In the Gentlemen's model the minimum number of comparisons needed to find six elements which contains the maximum and the minimum is at most* $(11/4 - 1/96)n + O(1)$.

In our last model an unlimited number of erroneous answers may occur, but every element may be involved in at most one erroneous comparison. We call this model the *1-factor model* (as the edges in the championship graph corresponding to the lies form a (partial) matching). As Claim 45 remains valid in this model, the trivial upper bound $11/4n$ of Theorem 49 holds and the lower bound $87n/32$ of Theorem 48 is also true. For the first thought, one might conjecture that in this model the trivial upper bound could be closer to the truth as there can be much more erroneous answers. Contrary to this, the following theorem holds.

Theorem 56 *In the 1-factor model the minimum number of comparisons needed to find the maximum and the minimum is* $87n/32 + \Theta(1)$.

Proof: The lower bound follows from Theorem 48. For the upper bound we have to describe an algorithm. We use again the potential function p introduced in Section 2. We will say that at a match we *gain c (or lose c)* if the sum $\sum p(x)$ decreases by $1+c$ (or $1-c$) at that match. Note that if teams x and y play such that $wl(x) = wl(y)$ then we do not lose or gain anything.

At the beginning of our algorithm, we pick 8 teams $x_1, x_2, x_3, x_4, y_1, y_2, y_3$, y_4 and x_i plays y_i for all $1 \leq i \leq 4$. We may suppose that the x_i's win and now x_1 plays x_2 and x_3 plays x_4. Finally the losers of these two matches play. We may assume that x_1 is the team that won its first match and lost the other two. Note that until now we did not lose or gain anything as at every match the wl-value of the playing teams were the same.

Now $wl(x_1) = (1, 0)$ and $wl(y_1) = (2, 1)$ and we replay their match. If x_1 wins again, then we gain $1/4$ and repeat this procedure with the next 8 teams. If this time y_1 beats x_1, then $wl(x_1)$ stays $(1, 0)$, while $wl(y_1)$ becomes $(1, 1)$, thus we lose $1/4$, but we know that any further match involving x_1 or y_1 will give the true result. To exploit this fact we pick 5 more teams u, v_1, v_2, w_1, w_2 with wl-value $(2, 2)$. Let y_1 play with u and v_i play with w_i for $i = 1, 2$. At the match between y_1 and u we gain $1/4$ as $wl(u)$ will be $(0, 2)$ or $(2, 0)$, since the result of this match cannot be a lie. At the matches between the v_i's and the w_i's we do not gain or lose anything, but then x_1 should play one of the losers (the team with wl-value $(2, 1)$) and y_1 should play the other loser if y_1 lost to u (i.e. $wl(y_1) = (1, 0)$) and with a winner if y_1 beats u. It is easy to verify that because these matches cannot have erroneous results, we will gain $1/4$ at each of these matches, thus in total we gain $3 \cdot 1/4 - 1/4 = 1/2$ at matches involving these 13 teams.

So we obtained that depending on the answer we got for the replay between x_1 and y_1, we can gain $1/4$ at matches involving 8 teams or $1/2$ at matches involving 13 teams. Therefore we can gain at least $n/8 \cdot 1/4 = n/32$ which gives the upper bound of the theorem, since we can finish the algorithm in such a way that until the last few matches every match is played between teams with the same wl-value. ■

Chapter 7

Concluding remarks

In Chapter 2 we determine the largest possible cardinality of chain-intersecting families. In [5] the following two properties were introduced.

Definition 8 *A family $\mathcal{F} \subseteq 2^{[n]}$ is called* ***strongly*** (p, q)***-chain-intersecting*** *if there are no sets $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_p$ and $B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq B_q$ in $\mathcal{F}$ such that $A_p \cap B_1 = \emptyset$ (the top of a chain of length p always intersects the bottom of a chain of length q in $\mathcal{F}$).*

Definition 9 *A family $\mathcal{F} \subseteq 2^{[n]}$ is called* ***totally*** (p, q)***-chain-intersecting*** *if there are no sets $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_p$ and $B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq B_q$ in $\mathcal{F}$ such that $A_1 \cap B_1 = \emptyset$ (the bottoms of two chains of sizes p and q in $\mathcal{F}$ always intersect).*

We were able to determine the maximum size in the case of strongly p, q-chain-intersecting families for some values of n, p and q, but the full solution seems to be much harder.

In Chapter 2 we determined the maximum size of the r-complementing-chain-pair-freefamilies as a tool to determine the maximum size of a chain-intersecting family. In Chapter 3 we were able to determine the profile polytope of the r-complementing-chain-pair-freefamilies. However, determining the profile polytope of the chain-intersecting families is subject of further research.

In Chapter 5 we investigated the profile polytope of intersecting families in the poset of subspaces.

In Chapter 4 we found the essential extreme points of the l-chain profile polytope of many classes of families. In fact, in most of the cases if the case $l = 1$ is solved, the case of general l is also solved. Most of the exceptions are from our Chapter 3. However, in these problems it seems to be much harder to determine the extreme points in the case $l > 1$.

In Chapter 5 we determined the profile polytope of the intersecting families in the poset of subspaces. The profile polytope of t-intersecting families has not yet been determined neither in the Boolean case nor in the poset of subspaces, but in both cases we know how large can be the ith coordinate of the profile for all $0 \leq i \leq n$.

Theorem 57 (Frankl - Wilson [18]) *t If $\mathcal{U} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ is a t-intersecting family and $n \geq 2k - t$, then*

$$|\mathcal{U}| \leq \max\{\begin{bmatrix} n-t \\ k-t \end{bmatrix}, \begin{bmatrix} 2k-t \\ k \end{bmatrix}\}.$$

The corresponding extremal families are

i, $\mathcal{U}_0 = \{U \in \begin{bmatrix} V \\ k \end{bmatrix} : T \subseteq U\}$ where T is a fixed t-dimensional subspace of V,

ii, $\mathcal{U}_1 = \begin{bmatrix} W \\ k \end{bmatrix}$ where W is a fixed $2k - t$-dimensional subspace of V.

Theorem 58 (Ahlswede - Khatchatrian [1]) *If $1 \leq t \leq k \leq n$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is a t-intersecting family, then*

$$|\mathcal{F}| \leq \max_{0 \leq r \leq \frac{n-t}{2}} |\mathcal{F}_r|,$$

where $\mathcal{F}_r = \{F \in \binom{[n]}{k} : |F \cap [1, t+2r]| \geq t + i\}$ for $0 \leq r \leq \frac{n-t}{2}$.

These two theorems show that in the case of subspaces the extremal family is always one of two candidates, while in the Boolean case (as n goes to infinity) there are arbitrary many candidates (in fact Theorem 58 in its full strength gives for all r the range of k where $\mathcal{F}_r$ is the extremal family). Therefore one may suspect that it can be much easier to determine the profile polytope in the lattice of subspaces, than determining it in the Boolean case.

In Chapter 6 we determined the number of questions needed to find both the maximum and minimum elements using pairwise comparisons, if one lie is allowed. The most important unsolved question is of course to find the minimum number of comparisons needed in the cases where $k > 1$ lies are allowed. How much better can one do than the simple upper bound of Theorem 49? We conjecture that $(k + 1 + \epsilon)m$ questions are enough, where ϵ goes to zero as k increases.

We considered only the adaptive version of the problem, where we can see the answers before the next question. The non-adaptive version, where all the questions have to be asked at the same time is trivial (all the $\binom{n}{2}$ possible questions have to be asked $2k + 1$ times). However the version where r rounds are possible might be interesting.

Finally, let us remark that in that chapter we considered problems in totally ordered sets. Finding analogous results for partially ordered sets can be object of future research.

Bibliography

[1] R. Ahlswede, L. Khachatrian, The complete intersection theorem for systems of finite sets, *European J. Combin.* **18**, (1997), 125-136.

[2] R. Ahlswede, Z. Zhang, An identity in combinatorial extremal theory, *Advances in Mathematics* **80**, (1990), no. 2, 137-151.

[3] M. Aigner, Finding the maximum and the minimum, *Discrete Applied Mathematics* **74** (1997), 1-12.

[4] A. Bernáth, On a problem in extremal set theory, *Mat. Lapok* (N.S.) 10 (2000/01), **2**, (2005), 2-4 (Hungarian).

[5] A. Bernáth, D. Gerbner, Chain intersecting families, *Graphs and Combinatorics*, **23**, (2007), 353-366.

[6] C. Bey, Polynomial LYM inequalities, *Combinatorica* **25**, (2005), 19-38.

[7] B. Bollobás, On complete subgraphs of different order, *Math. Proc. Camb. Philos. Soc.*, **80** (1976) 19-24.

[8] C. Deppe, Coding with Feedback and Searching with Lies, in *Entropy, Search, Complexity*, Bolyai Society Mathematical Studies, **16** (2007), 27-70.

[9] K. Engel, Sperner Theory, Encyclopedia of Mathematics and its Applications, 65. Cambridge University Press, Cambridge (1997)

[10] K. Engel, P.L. Erdős, Sperner families satisfying additional conditions and their convex hulls, *Graphs Combin.* **5**, (1989), 47-56.

[11] K. Engel, P.L. Erdős, Polytopes determined by complementfree Sperner families, *Discrete Mathematics* **81**, (1990), 165-169.

[12] P. Erdős, On a lemma of Littlewood and Offord, *Bull. of the Amer. Math. Soc.* **51**, (1945), 898-902.

[13] P. Erdős, C. Ko, R. Rado, Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford* Ser. 2 **12**, (1961), 313-318.

[14] P.L. Erdős, P. Frankl, G.O.H. Katona, Intersecting Sperner families and their convex hulls, *Combinatorica* **4**, (1984), 21-34.

[15] P.L. Erdős, P. Frankl, G.O.H. Katona, Extremal hypergraph problems and convex hulls, *Combinatorica* **5**, (1985), 11-26.

[16] P. Frankl, The convex hull of antichains in posets, *Combinatorica* **12(4)**, (1992), 493-496.

[17] P. Frankl - G.O.H. Katona, Polytopes determined by hypergraph classes, *European J. Combin.* **6** (1985), no. 3, 233-243.

[18] P. Frankl - R.M. Wilson, The Erdős-Ko-Rado theorem for vector spaces, *J. Combin. Theory* Ser. A **43**, (1986), no. 2, 228-236.

[19] D. Gerbner, On an extremal problem of set systems. *Mat. Lapok* (N.S.) 10 (2000/01), no. 2, (2005), 5-12 (in Hungarian).

[20] D. Gerbner, Profile polytopes of some classes of families, *in preparation*

[21] D. Gerbner, D. Palvolgyi, B. Patkós, G. Wiener, Finding the maximum and minimum elements with one lie, *Disc. Appl. Math.*, to appear

[22] D. Gerbner - B. Patkós, l-chain profile vectors, *SIAM Journal on Discrete Mathematics* **22** (2008), 185-193.

[23] D. Gerbner, B. Patkós, Profile vectors in the poset of subspaces, *Discrete Mathematics*, *Discrete Mathematics* **309**, (2009), 2861-2869.

[24] C. Greene - D.J. Kleitman, Proof techniques in the theory of finite sets, *Studies in combinatorics*, MAA Stud. Math., **17**, Math. Assoc. America, Washington, D.C., (1978), 22–79.

[25] W.N. Hsieh, Intersection theorems for systems of finite vector spaces, *Discrete Math.* **12**, (1975), 1-16.

[26] G.O.H. Katona, A simple proof of the Erdős-Chao Ko-Rado theorem, *J. Comb. Theory (B)*, **13** (1972), 183-184.

[27] G.O.H. Katona, The cycle method and its limits, *Numbers, information and complexity (Bielefeld, 1998)*, Kluwer Acad. Publ., Boston, MA, (2000) 129-141.

[28] D. Lubell, A short proof of Sperner's lemma, *J. Combinatorial Theory* **1**, (1966), 299.

[29] L.D. Meshalkin, Generalization of Sperner's theorem on the number of subsets of a finite set, *Teor. Veroyatn. Primen.*, **8** (1963), 219-220.

[30] E.C. Milner, A combinatorial theorem on systems of sets, *J. London Math. Soc.*, **43**, (1968), 204-206.

[31] B. Patkós, AZ-type identities, internet note, downloadable from http://www.ceu.hu/math/Preprints/Patkos3.pdf

[32] A. Pelc, Searching games with errors – Fifty years of coping with liars, *Theoretical Computer Science* **270** (2002), 71-109.

[33] I. Pohl, A sorting problem and its complexity, *Communications of the ACM* **15** (1972), 462-464.

[34] G. Purdy, A result on Sperner collections, *Utilitas Math.*, **12**, (1977), 95-99.

[35] B. Ravikumar, K. Ganesan, K.B. Lakshmanan, On Selecting the Largest Element in Spite of Erroneous Information, STACS 1987, 88-99.

[36] A. Sali, A Note on Convex Hulls of More-part Sperner Families, *J. Combin. Theory* Ser. A **49**, (1988), 188-190.

[37] N. Sauer, A generalization of a theorem of Turán, *J. Combin .Theory* Ser. B **10**, (1971), 109–112.

[38] Sperner, Ein Satz über Untermengen einer endlichen Menge, *Math. Z.* **27**, (1928), no. 1, 544–548 (German).

[39] P. Turán, An extremal problem in graph theory, *Mat. Fiz. Lapok*, **48**, (1941), 436-452 (Hungarian).

[40] K. Yamamoto, Logarithmic order of free distributive lattice, *J. Math. Soc. Japan* **6** (1954), 343-353.